PHILOSOPHICAL TRANSACTIONS.

I. *A Memoir on the Theory of Mathematical Form.**

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Received May 18,—Read June 18, 1885.

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* In accordance with criticisms and suggestions of the referees, an alteration has been made in the title of the memoir, and certain sections towards the end have been omitted. There is consequently a slight want of complete correspondence between the memoir and the abstract given in the Proceedings, vol. 38, p. 393.—A. B. K.

Scope of the Memoir.

1. My object in this memoir is to separate the necessary matter of exact or mathematical thought from the accidental clothing—geometrical, algebraical, logical, &c.—in which it is usually presented for consideration; and to indicate wherein consists the infinite variety which that necessary matter exhibits.

2. The memoir is confined to the exposition of fundamental principles, to their elementary developments, to their application to such a variety of cases as will vindicate their value, and to a description of some simple and uniform modes of putting the necessary matter in evidence. I have been unable to ascertain that the principles here set forth have been previously formulated.

Fundamental Principles.

3. Whatever may be the true nature of things and of the conceptions which we have of them (into which points we are not here concerned to inquire), in the operations of reasoning they are dealt with as a number of separate entities or *units.*

4. These units come under consideration in a variety of garbs—as material objects, intervals or periods of time, processes of thought, points, lines, statements, relationships, arrangements, algebraical expressions, operators, operations, &c., &c., occupy various positions, and are otherwise variously circumstanced. Thus, while some units are incapable of being distinguished from each other, others are by these peculiarities rendered distinguishable. For example, the angular points of a square are distinguishable from the sides, but are not distinguishable from each other. In some instances where distinctions exist they are ignored as not material. Both cases are included in the general statement that some units are distinguished from each other and some are not.

5. In like manner some *pairs* of units are distinguished from each other, while others are not. Pairs may in some cases be distinguished even though the units

composing them are not. Thus the angular points of a square are undistinguishable from each other, and a pair of such points lying at the extremities of a side are undistinguishable from the three other like pairs, but are distinguished from the two pairs formed by taking the angular points at the extremities of the diagonals, which pairs again are undistinguishable from each other. Further, a pair ab may sometimes be distinguished from the pair ba though the units a and b are undistinguished. Thus, in fig. 1 the three black spots a, b, c, are all undistinguished from each other, each has an

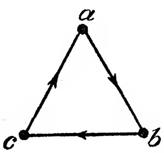


Fig. 1.

arrow proceeding to it and an arrow proceeding from it, but the pair ab is distinguished from the pair ba, for an arrow proceeds from a to b, but none from b to a.

6. It will be convenient to speak of ab and ba as different *aspects* of the *collection* of two units a, b. Here the terms "aspect" and "collection" are each to be understood as referring to two separate units, and not to those units regarded in the aggregate as a single unit.

7. Again, there are also distinguished and undistinguished *triads*, *tetrads*, . . . *m-ads*, . . . *n-ads* . . . ; every m-ad being, of course, distinguished from every n-ad. Just as we may have ab distinguished from ba, though a is undistinguished from b, so we may have an n-ad $pqrst \ldots uv$ distinguished from $qusvt \ldots rp$, though the units p, q, r, s, &c., are all undistinguished from each other, and further, though their pairs are also undistinguished, as likewise their triads, &c. Here $pqrst \ldots uv$ and $qusvt \ldots rp$ will be termed, as in the case of pairs, different *aspects* of the *collection* $p, q, r, s, t, \ldots u, v$; the term "collection" being understood to refer to a number of separate units without reference to the various "aspects" of the collection. Different aspects of the same collection of n units will be regarded as different n-ads.

8. The terms "pair," "triad" . . . "n-ad," "collection," "aspect" will always be understood to refer to two, three, n, &c., units, and never to aggregations of units considered as a single unit. Pairs, triads, n-ads, collections, aspects may, of course, be regarded as units, but when they are so regarded the fact will be distinctly pointed out.

9. Every collection of units has a definite *form* due (1) to the number of its

Fig. 2.

component units, and (2) to the way in which the distinguished and undistinguished units, pairs, triads, &c., are distributed through the collection. Two collections of

the same number of units but having different distributions will be of different forms. Thus the two tetrads a, b, c, d, and p, q, r, s, of fig. 2, contain the same number (four) of unit spots, but they are of different forms; for a, b, c, d are all undistinguished from each other; while q, r, s, though undistinguished from each other, are all distinguished from p. The distribution of the distinguished and undistinguished pairs and triads is also obviously different in the two cases. The word "form" will in this memoir be always employed in the sense here indicated.

10. Two collections of units which are undistinguished are of the same form, but two collections which are of the same form are not necessarily undistinguished; there may be the same distribution of distinguished and undistinguished units, pairs, &c., in each, while the units, pairs, &c., of one are all distinguished from the units, pairs, &c., of the other.

11. Each of the forms which a system of any number n of units can assume, owing to varieties of distribution, is one of a definite number of possible forms, and the peculiarities and properties of the collection depend, as far as the processes of reasoning are concerned, upon the particular form it assumes, and are independent of the dress, geometrical. algebraical, logical, &c., in which it is presented; so that two systems which are of the same form have precisely the same properties, although the garbs in which they are severally clothed may, by their dissimilarity, lead us to place the systems under very different categories, and even to regard them as belonging to "different branches of science."

12. It may seem in some cases that other considerations are involved besides "form," but it will be found on investigation that the introduction of such considerations involves also the introduction of fresh units, and then we have merely to consider the form of the enlarged collection.

13. In order to put form in evidence some "accidental" clothing is of course necessary; if, however, we employ more than one species of clothing, each species being uniform and suited to forms of every kind, the likelihood of its accidental nature being overlooked will be reduced to a minimum.

Units.

14. The units which we have to consider exhibit endless variety; thus we may have a material object dealt with as one unit, a quality it possesses as another, a statement about it as a third, and a position it occupies in space as a fourth. The task of specifying the units which are considered in an investigation may in some cases be one of considerable difficulty, and mistakes are likely to occur unless the operation is conducted with great care.

15. We have frequently to deal with things x, y, z, &c., pairs, &c., of those things, the differences between which depend on the existence or non-existence of certain circumstances, or upon taking into account or ignoring certain circumstances. Thus we may apparently have collections of units x, y, z, &c., which are at one time t, when

one state of things exists, of one form, and at another time t' of another form. This is, however, not so; the form of a collection is absolutely invariable. The apparent alteration of form arises from overlooking the fact that the units dealt with are not x, y, z, &c., but (x at time t), (y at time t), &c., (x at time t'), (y at time t'), &c. The unit (x at time t) may be quite different from (x at time t'). Where there is only one alternative t, the collection (x at time t), (y at time t), (z at time t), &c., is of the same form as the collection x, y, z, &c., and it matters not which is the collection dealt with.

16. Aggregations of those things which, in a symbolical representation of the units considered in any case, are already represented as units, must not be supposed to be sufficiently represented without additional symbols, but must each be represented by a distinct symbolical unit. An aggregation of things is, as far as the processes of reasoning are concerned, a mere unit, and must be so dealt with.

17. While it is important that a unit should be represented and dealt with as a unit, it is equally important that we should not be misled by our modes of thought and consequent use of language into regarding a number of distinct units as one only. A collection of units must in a symbolical representation be represented as a single unit where it is so regarded; but where the word "collection" is used, as it is here (sec. 8), merely to "denote" or mark off a number of things each of which is considered as a distinct unit, we must be careful to represent each of those things by a distinct symbol.

Some Definitions.

18. Any collection of units which consists entirely of units selected from another collection will be termed a *component* of the latter. Any aspect of a component of a collection may also be spoken of as a component of the collection.

19. Two collections of units will be said to be *detached* if they have no component in common.

20. An n-ad which has one or more units in common with each of a number of collections will be said to be an n-ad *connecting* those collections; *e.g.*, the pairs which a single unit makes with the various units of a collection will be termed the pairs connecting the unit and the collection. An n-ad connecting n detached collections has one unit and one only in common with each. If A, B, C, D, ... be collections of which a, b, c, d, . . . are component units respectively; when an n-ad connecting A B C D . . . is spoken of, it must be understood that an n-ad such as $a\ b\ c\ d$ . . . is meant, and not one such as $b\ d\ c\ a$ . . ., which will be spoken of as connecting B D C A . . .

21. If the component units of a collection are all undistinguished from each other the collection will be said to be *single*.

22. Units which are undistinguished from the same unit are undistinguished from each other; thus if a collection of units is not single it consists of two or more detached single collections, and will be termed a *double*, *treble*, *&c.*, collection, according to the number of component single collections which it contains.

23. Just as collections of units break up into single collections of undistinguished units, so collections of pairs break up into single collections of undistinguished pairs, and, generally, collections of n-ads into single collections of undistinguished n-ads.

24. Differences between the various units, pairs, &c., of a collection may be termed the *internal* differences of that collection. Differences between the various units, pairs, &c., of a collection A and the units, pairs, &c., of a detached collection may be said to be *external* to A.

Systems.

25. If every component unit of a collection is distinguished from every unit which is detached from the collection, the collection will be termed a *system.*

26. Every collection of units is a component of some system.

27. The whole collection of units which come under consideration in any inquiry is a system; for the units are distinguished from all others by being the only ones considered.

28. The n-ads of one single system of units are distinguished from all those of any other single system of units, and from all the n-ads connecting any two systems of units. The connecting n-ads of any n systems of units are distinguished from those of any other n systems of units, and themselves break up into single systems of n-ads.

29. The units of a single system of units must be dealt with as a whole; for, as they are undistinguished from each other, no definition can be given of, or remark made about, one which is not equally applicable to each of the others. Each can only be spoken of as "one of" the units of the system. A similar observation applies in the case of single systems of pairs, triads, &c.

30. Many systems of units are defined by stating that certain of their components constitute a system. It must be borne in mind that such a statement does not mean that the components are undistinguished from each other, but merely that they are distinguished from all others; the system of components may be a multiple one. It is by no means unnecessary to emphasize this, as we are prone to assume that the mode of classifying things is that of putting like things into the same class, rather than that of putting unlike things into different classes.

31. The distribution of the various distinguished and undistinguished components of a system is regulated by definite laws; so that a knowledge of the mode of distribution of some only of the distinguished and undistinguished components may determine the form of the system. There are in general several ways in which the form of a given system may be thus determined, and accordingly various different definitions of the same system may be adopted.

32. For the statement of some of the properties of a system S it may be necessary to have the form fully defined; in the case of others this may not be necessary, it being sufficient to state that certain components are distinguished, without asserting anything as to others, *i.e.*, to state that certain components constitute a system,

without asserting whether the system be single or multiple. The result of this may of course be that we deal with different units at different stages of the investigation of S (secs. 126 *ff*).

33. Systems are frequently more readily dealt with when regarded as components of more extensive systems. Much ingenuity has been expended on the discovery of systems, the addition of which to the particular system under consideration may assist in its investigation. It is to this that the existence of such units as substitutions, quarternions, quadrates, &c. (as to some of which I shall have to speak presently), is due. We shall see that we can always, by the addition of a proper system to any given system, completely define the latter by merely indicating the mode of distribution of certain pairs (secs. 81, 82).

34. Notwithstanding the great assistance derived from the use of added systems, much reluctance is exhibited in employing them unless they can be shown to have their representatives in nature, *i.e.*, unless "accidental" clothing can be found to fit them. The objections raised to symbolical methods which cannot be "interpreted" are strong evidence of the fact that the accidental nature of much that comes under our consideration is not really appreciated.

35. It is by no means unnecessary to state that the form of a system is independent of the particular method of defining it which we adopt; and that because it is easier to define a system by adding fresh units, it does not owe its form to the existence of those units.

36. The system which is the actual subject of investigation in an inquiry may be termed the *base* system; those systems which are added for the purpose of assisting in the investigation being termed *auxiliary* systems.

Heaps.—Graphical Representation of Units.

37. There are two forms of systems of n units the consideration of which properly precedes that of all others. The one is that of a system which consists of n units, each of which is distinguished from each of the others, so that every component s-ad is distinguished from every other component s-ad for all values of s from 1 to n. A system of this form I shall term a *discrete heap.*

38. The other is that of a system of n units which is such that every component s-ad is undistinguished from every other component s-ad for all values of s from 1 to n. A system of this form I shall term a *single heap.*

39. A discrete heap may be graphically represented by a number of small separated

Fig. 3.

circles each containing a different letter (fig. 3). The letters render the *graphical units* distinguishable from each other.

40. A single heap may be graphically represented by a number of small separated circles each containing the same letter. (Fig. 4.)

Fig. 4.

41. Differences arising from the positions of the graphical units on the paper on which they are drawn are always to be disregarded. Where positions are considered, they must be represented by separate graphical units.

42. Where there is occasion to employ letters outside and adjacent to the graphical units for purposes of reference, differences arising from the use of such letters are also to be ignored. (Sec. 173.)

43. In every system of n units other then a discrete or single heap, some s-ads are distinguished from each other and some are not, for all or some values of s from 1 to n. Thus every form which a system of n units can assume may be regarded as either that of a discrete or single heap or as intermediate between the two.

44. There are systems of n units of other forms than those of discrete and single heaps which can be represented by means of graphical units alone. These consist of one or more detached independent single heaps. The term "independent" will be fully explained presently (secs. 117, 118); for the present it will be sufficient to state that two independent systems are such as can be graphically represented on separate sheets of paper, without the employment of symbols on either sheet relating to those on the other.

45. A system of s independent detached heaps will be termed an s-tuple heap. Fig. 5 represents a treble heap of seven units.

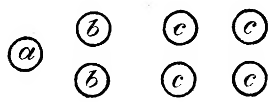


Fig. 5.

46. A discrete heap of n units is an n-tuple heap of n units.

47. The number of different forms which a heap of n units can assume is the same as that of the partitions of n.

48. It will in some cases be convenient, instead of employing circular graphical units distinguished by internal letters, to use coloured graphical units, or black spots of different sizes, or graphical units of various shapes.

49. Where graphical units are employed alone, like units will represent undistinguished units, but in the case of forms other than heaps it will be necessary to employ means to distinguish pairs, &c., and these may cause like graphical units to be distinguished from each other. No general inference must therefore be drawn from the use of like and unlike graphical units other than that unlike units represent distinguished units.

Pairs.—Graphical Representation.

50. There are three different forms of pairs; viz., if a, b be two units, we can have :—

(1) a distinguished from b, and therefore also ab distinguished from ba;

(2) a undistinguished from b, but ab distinguished from ba;

(3) a undistinguished from b, and ab undistinguished from ba.

51. The two units in case (1) belong to different single systems, and thus the pair is a pair connecting two single systems.

52. In cases (2) and (3) both units belong to the same single system, and the pairs are components of that system. The units in case (2) may be said to form an *unsymmetrical* pair, those in case (3) being said to form a *symmetrical* pair. It will be convenient in speaking of undistinguished unsymmetrical pairs ab, cd, to say that ab and cd are *of the same polarity*, and that ab and dc are *of opposite polarities*. Of course in this case ab and ba are of opposite polarities. In the case of a symmetrical pair e, f, ef and fe will be of the same polarity.

53. The expression "unsymmetrical" may also be applied to pairs falling under case (1). We may also in the case of such pairs say that ab and cd if undistinguished are of the same polarity, ab and dc of opposite polarities.

54. If in a diagram consisting of a number of graphical units, some pairs of those units are joined by plain lines or links, as in fig. 6, while others are not so joined, the

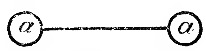


Fig. 6.

pairs will be divided into two systems. Pairs which are thus joined by links will not be necessarily undistinguishable, nor will pairs which are not joined by links be necessarily undistinguishable, but we merely have pairs which are joined by links distinguished from pairs which are not so joined. Thus the two systems into which the pairs are divided may each be either a single or multiple system.

55. A large number of systems may be completely defined by diagrams consisting of graphical units and links only. We have already seen that an extensive class

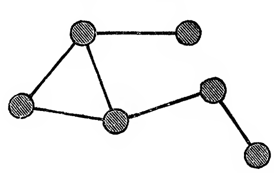


Fig. 7.

of systems may be represented by diagrams consisting of different sorts of graphical units only; there are also systems which may be fully defined when we indicate by links a dichotomy of their component pairs, without defining further what units are distinguished and what are not. Thus the system of fig. 7 must be a discrete

heap, each graphical unit admitting of separate definition. By the use of links in conjunction with graphical units of different sorts the number of representable forms is greatly increased. I shall give in the following four sections some examples of forms which may be represented by the use of graphical units and links.

56. Figs. 8 and 9 illustrate the fact that a system may be defined in various ways;

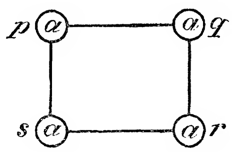


Fig. 8.

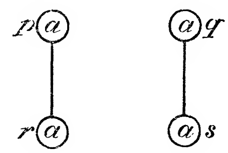


Fig. 9.

they represent systems of the same form. Pairs which are not joined by links in fig. 8 are joined in fig. 9, and *vice versâ*.

57. I give the systems of figs. 10 and 11 as examples of two varieties of single

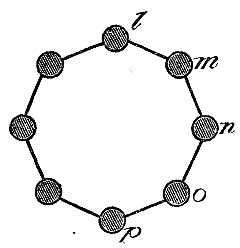


Fig. 10.

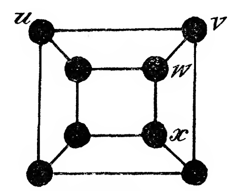


Fig. 11.

systems of the same number of units. In fig. 10 there are four sorts of pairs of which *lm*, *ln*, *lo*, *lp*, are types. In fig. 11 there are only three sorts, of which *uv*, *uw*, and *ux*, are types. It may be noticed that in each case the linked pairs compose a single system; while the unlinked pairs in the former case compose a treble system, in the latter a double one.

58. Fig. 12 is an example of a triple system containing unsymmetrical pairs of both

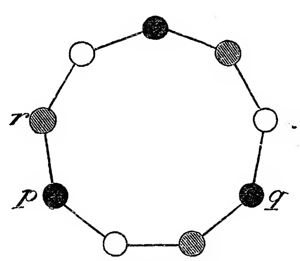


Fig. 12.

sorts. We have *p*, *q* an unsymmetrical pair composed of two undistinguished units, and *p*, *r* an unsymmetrical pair composed of two distinguished units.

59. The single system shown in fig. 13 is one of considerable interest; it is that dealt with in the case of the theorem that if two coplanar triangles are coaxial they

are also copolar (Theorem (1), sec. 357). The graphical units may be taken to represent either the ten straight lines of the theorem, or the ten points of intersection;

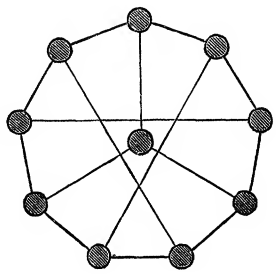


Fig. 13.

the form is the same in either case. Taking the former case, the pairs of graphical units which are joined by links correspond to pairs of lines whose points of intersection are points other than the ten considered in the theorem. It should be noticed that the two systems into which the pairs are divided (linked and unlinked), are each single.

60. We shall see (secs. 81, 82), that by the addition of units to any system S, *i.e.*, by regarding S as a component of a more extensive system, we can represent the form of S by the use of graphical units and links only. It will, however, be convenient in many cases to employ some further devices which will enable us, without employing additional graphical units, to represent forms which could not be exhibited by the use of plain links only.

61. Thus we may have lines of various sorts joining two graphical units, viz., dotted, wavy (fig. 14), red, blue, &c., in addition to links, which will always be under-

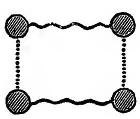


Fig. 14.

stood to be plain lines. Pairs joined by unlike lines will be, as in the case of links, distinguished from each other.

62. Where a pair is unsymmetrical an arrow-head or barb may be added to the link or other line joining the two graphical units which represent the pair (fig. 15). The arrow-head has the effect of making ab distinguished from ba. In the fig. the arrow-

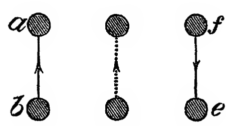


Fig. 15.

heads make ab distinguished from fe, but not from ef. The relative directions of the arrow-heads in the case of pairs joined by unlike lines are immaterial. It is

unnecessary to add an arrow-head in the case of a pair of distinguished units; the difference between the graphical units indicates the absence of symmetry.

63. We may in some cases of symmetrical pairs instead of employing a single line, use two barbed lines of opposite polarities, as in fig. 16.

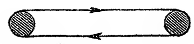


Fig. 16.

64. In place of using a number of different sorts of lines, we may use plain lines, barbed or unbarbed according as the pairs are unsymmetrical or symmetrical, with letters alongside of them; like letters being used where otherwise like lines would be used, and unlike where unlike lines would be used (fig. 17). Though this method is not so good as the previous one as regards its power of enabling us to visualize the

Fig. 17.

systems represented, it has distinct advantages for purposes of description, as we can denote pairs of different sorts by the annexed letters, the sorts (sec. 84) of which denote the sorts of the pairs.

65. If σ denotes an unsymmetrical pair ab, we may denote the pair ba by $\rotatebox{180}{$\sigma$}$; or, as this is in some instances awkward, by σ' where $(\sigma')'=\sigma$.

66. Although, as we have seen, it is in many cases unnecessary to draw lines connecting all pairs of graphical units in order to completely define a system (secs. 55–59), it may in some cases be desirable to do so, especially where we wish to show how many different sorts of pairs there are under consideration. In such cases we may connect like pairs by like lines and unlike by unlike lines. Thus the diagram of fig. 18, which completely defines a system of six units, may be completed as in

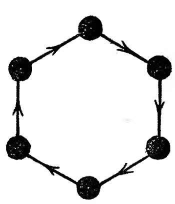


Fig. 18.

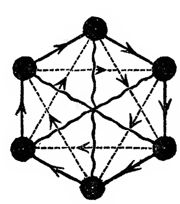


Fig. 19.

fig. 19 so as to show that there are two sorts of unsymmetrical pairs and one sort of symmetrical pairs.

67. In general, in diagrams in which lines other than links are employed, either in conjunction with links or not, like pairs of graphical units will be joined by like lines.

68. It can readily be shown that every form which admits of representation by

means of graphical units and lines of different sorts, can, by the introduction of additional or *auxiliary* graphical units, be represented by means of graphical units and links only. It will be convenient for the purpose of description to suppose that the lines of different sorts are of different colours. In the place of any red line substitute a red unit (*i.e.*, one coloured red, instead of having a letter inside), this unit being connected by two links to the two units which were joined by the red line (fig. 20). Effect this change in the case of all the red lines; and

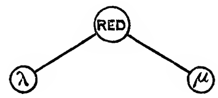


Fig. 20.

similarly in the place of blue, yellow, &c., lines, substitute blue, yellow, &c., units, joined by links to the units which were connected by the several coloured lines. The red units and links will have precisely the same effect in rendering pairs distinguishable as the red lines had. We may, if we please, substitute letters for the various colours in the added units, and so obtain a diagram consisting of graphical units and links, representing a collection of which the original one is only a component. This component is, however, of the same form as before.

69. In the case of an unsymmetrical pair, pq, of undistinguished graphical units which would ordinarily be joined by a barbed line, we must have two additional undistinguished graphical units in some cases, as in fig. 21, where u, v are the auxiliary units.

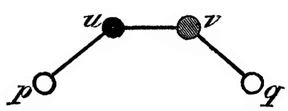


Fig. 21.

70. In the case of unsymmetrical pairs of distinguished units we have seen that no barbs are necessary (sec. 62), and we need therefore only one additional graphical unit, as in sec. 68.

71. All systems of which the forms are determined by the distinguishableness and undistinguishableness of units and pairs only, may be represented by the methods we have been considering. But there are also systems, the forms of which are not so determined, but depend upon the distinguishableness and undistinguishableness of triads, tetrads, &c. Such systems may at first sight require other expedients for their graphical representation, but we shall see, as already stated (secs. 33, 60), that these are not necessary.

72. Diagrams such as those we have been discussing may come under our consideration in other ways than as representing graphically systems of different forms. In such cases it must not be assumed that the units under consideration

are the small circles, or only those; in general this will not be so; the units may be the links, or the *ways* in which a number of small circles or spots can be connected by links, &c. If we ascertain exactly what the units really dealt with are, we may construct a graphical representation of the system they constitute, and this may be of a very different character from that of the diagram from which it is derived.

Aspects.

73. If two collections of units are undistinguished, they may be regarded as corresponding to each other in one or more ways, in each of which correspondences to each unit, and therefore to each pair, triad, &c., of one collection there corresponds in the other a counterpart unit, pair, triad, &c., undistinguished from the former in dress or other circumstance. In any one of these correspondences, two corresponding units are regarded as occupying corresponding places, or, as we may express it, places of the same *sort*. Furthermore, each of the corresponding units is regarded as belonging to one or the other of the two collections, each of which is regarded also in the aggregate as a single unit.

74. Now the unit A which is dealt with when we thus regard a unit a as occupying a place of a particular sort g in a particular collection, is a different unit from a; it may be called an *aspect* of a.

75. Thus when we consider a correspondence of two undistinguished collections $a, b, c, d, \ldots$ and $p, q, r, s, \ldots$ where

a corresponds with p
b ,, ,, q
c ,, ,, r

and so on, we deal with a collection of units A, B, C, D, ... P, Q, R, S, ... where any unit R is the unit arrived at by considering the unit r as occupying a place of a particular sort, the unit C being that arrived at by considering the unit c as occupying a place of the same sort; and similarly in the case of the other corresponding units. Each of the collections A, B, C, D, . . . and P, Q, R, S, ... is an *aspect*, the former of the collection $a, b, c, d, \ldots$ and the latter of the collection $p, q, r, s, \ldots$ We also regard A, B, C, D, . . . in the aggregate as a single unit V, which may be termed a *unified aspect* of $a, b, c, d, \ldots$ and so also in the case of P, Q, R, S, . . .

76. The collections A, B, C, D, . . . and P, Q, R, S, . . . may be different aspects of the same collection $l, m, n, o, \ldots$, as a collection may be *self-correspondent*, and the number of units $a, b, c, d, \ldots$ be accordingly the same as that of the number of sorts of places considered.

77. We may have a number of undistinguished collections, each of n units, all corresponding to each other. If there be m such collections, in any correspondence in

which they all correspond we have mn unit aspects such as A, n sorts of places, and each sort of place occupied by m units.

78. A, the aspect of a in a correspondence, bears definite relations (sec. 79) to three other units, viz.:—

(1.) a of which it is an aspect.

(2.) The unified aspect V arrived at by regarding as a unit the aspect of which it is a component.

(3.) The sort g of the place a occupies in the correspondence.

79. The "definite relations" of the preceding section consist merely in this, viz., that the pairs which A makes with a, V, and g, are all distinguished from all the other pairs which A makes with other units. (See secs. 143 *ff*, on *Associates.*)

80. The tetrad A, a, V, g, may be graphically represented as in fig. 22; the graphical unit A being joined to no other graphical units besides a, V, and g. Since

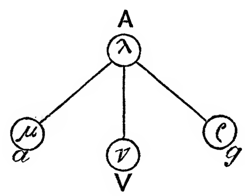


Fig. 22.

the units A, a, V, and g, are obviously distinguished from each other, the graphical units are made so also.

81. It appears, then, that in studying the form of a system S by means of correspondences of undistinguished components, we really regard S as a component of a more extensive system, containing

(1.) the system S;

(2.) the system X composed of units which are conceived of as sorts of places;

(3.) the system Y composed of units which are conceived of as unified aspects of all the various component collections of S;

(4.) the system Z composed of units which are conceived of as aspects of single units of S.

The pairs connecting Z and the joint system S, X, Y, break up into two systems, graphically represented by linked and unlinked pairs of graphical units respectively.

82. A graphical representation in which the units of S, X, Y, Z, are all represented by graphical units, and links connecting the graphical units are drawn in the manner indicated in the last two sections, fully defines the form of S, as it completely indicates what components are undistinguished and what are not. Such a graphical representation is sometimes termed a "linkage," so that we may say that every system may be graphically represented by a linkage (secs. 194, 195).

83. The system Y was described in sec. 81 as consisting of unified aspects of *all* the component collections of S, *i.e.*, the component pairs, triads, &c. It is, however, unnecessary so far as defining the form of S is concerned that this should be so. We shall see that it is sufficient that Y should include a unified aspect V of the whole system S, and all other unified aspects of S which are undistinguished from V (secs. 194, 195).

Letters, their Sorts and Positions.

84. Let us now turn aside to consider certain points with regard to the representation of units by letters. Any letter admits of being repeated as often as we please; each repetition being one of a number of letters of the same *sort*. When in common parlance we speak of letters a, b, c, d, &c., we are really referring, not to the letters themselves, but to their sorts. Each letter (not sort of letter) occupies a different position on the paper on which it is written or printed; as in the case of the letters, we have a number of positions of the same sort, each called "the same position" with reference to another of the number. For example, if we have two arrays of the same number of letters, *e.g.*, $abcde$ and $pqrst$, we say that "b occupies the *same position* in the first array that q does in the second," meaning that the sort of the position b occupies is the sort of that which q occupies.

85. The sorts of the letters and the sorts of the positions are both dealt with as units. Further, we regard each letter as belonging to a particular collection regarded in the aggregate as a single unit, say a *unified collection.*

86. Thus a letter bears definite relations to

(1.) its sort;

(2.) the unified collection to which it belongs;

(3.) the sort of its position.

Representation of Aspects of Collections by Arrays of Letters.

87. Thus we have an exact copy in a different dress of the state of things considered in secs. 73–83. If then we represent

(1.) the units of any collection by the sorts of letters; we may represent

(2.) the sorts of places occupied by those units in a correspondence with an undistinguished collection by the sorts of the positions of the letters;

(3.) an aspect of the collection by a collection of letters in sorts of positions;

(4.) unified aspects by unified collections of letters;

(5.) aspects of single units by single letters.

88. Taking, then, arrays as our collections of letters, an aspect may be represented thus $abcdef$ . . . . . (*i.e.*, in the manner adopted in sec. 7), where the order of the letters is to be regarded as material to this extent, viz., that though the aspect which is

represented by *abcdef* . . . . might equally well be represented by *bdacfe* . . . . , yet, if this be done, the aspect originally represented by *bdacfe* . . . must be represented by *dcbaef* . . . , and that originally represented by *pqrstu* . . . by *qsprut* . . . . , and similarly in the case of other aspects and other transpositions of the sorts of the positions occupied by the letters in the arrays. Whatever change be effected in the order of the letters in one array, a precisely similar change must be made in the order of the letters in each of the other arrays of the same number of letters. The relative order in arrays of different numbers of letters is immaterial, as such arrays represent collections of different numbers of units which are therefore distinguished, and are not regarded as corresponding.

Elementary Properties of Aspects.

89. The symbols $>\!\!-\!\!\!-\!\!<$ and $<\!\!-\!\!\!-\!\!>$ will be used to represent "is undistinguished from" and "is distinguished from" respectively; and such expressions as "let $a >\!\!-\!\!\!-\!\!< b$" must be read, "let a be undistinguished from b"; and similarly in other cases.

90. Symbols, such as $a, b, c, d,$ in which commas separate the letters, will always be supposed, as heretofore, to represent a collection of units without reference to particular aspects, the order of the letters being accordingly supposed to be immaterial.

91. A statement such as *abcd* . . . . $>\!\!-\!\!\!-\!\!<$ *pqrs* . . . . implies that the components represented by taking corresponding letters on each side of the $>\!\!-\!\!\!-\!\!<$ are undistinguished; *e.g.*, $bc >\!\!-\!\!\!-\!\!< qr$. A statement such as *abcd* . . . . $<\!\!-\!\!\!-\!\!>$ *pqrs* . . . . does not imply that $a, b, c, d,$ . . . . $<\!\!-\!\!\!-\!\!>$ $p, q, r, s,$ . . . . for it is consistent with the statement that *abcd* . . . $>\!\!-\!\!\!-\!\!<$ *srqp* . . . which implies that $a, b, c, d,$ . . . $>\!\!-\!\!\!-\!\!<$ $p, q, r, s,$ . . .

92. If *abcd* . . . $>\!\!-\!\!\!-\!\!<$ *pqrs* . . ., then *bdca* . . . $>\!\!-\!\!\!-\!\!<$ *qsrp* . . . ; and similarly in the case of any other transposition of letters.

93. If *pqrs* . . . be merely *abcd* . . . in a different order, so that we are considering two aspects of the same collection, and if we repeat on *pqrs* . . . the transposition by which it is derived from *abcd* . . ., we get another aspect which by the preceding section is undistinguished from *pqrs* . . ., and therefore from *abcd* . . .

94. If *abcd* . . $>\!\!-\!\!\!-\!\!<$ *pqrs* . ., and if $l, m, n, o,$ . . . be units other than $a, b, c, d,$ . . ., there must be units $w, x, y, z,$ . . other than $p, q, r, s,$ such that *abcd* . . . *lmno* . . . $>\!\!-\!\!\!-\!\!<$ *pqrs* . . . *wxyz* . . .

95. If *abcd* . . . $<\!\!-\!\!\!-\!\!>$ *pqrs* . . . each collection having the same number of units, and if $l, m, n, o,$ . . . be units other than $a, b, c, d,$ . . . there cannot be any units $w, x, y, z,$ . . . such that *abcd* . . . *lmno* . . $>\!\!-\!\!\!-\!\!<$ *pqrs* . . . *wxyz* . . . .

96. Every collection of n units has $\underline{|n}$ aspects. If m aspects of the collection are undistinguished from each other, but are distinguished from all other aspects of the

collection, m must be a factor of $\underline{|n}$, and the aspects break up into $\frac{|\underline{n}}{m}$ systems, each consisting of m aspects.

97. It follows immediately from sec. 92 that if we write down the symbols *abcd* . . ., &c., representing any aspect and all other aspects which are undistinguished from it, and we transpose the letters of *abcd* . . . and make precisely the same transpositions in the case of the letters of the symbols representing the other undistinguished aspects, we shall get a collection of symbols of undistinguished aspects which may or may not be the same as the former.

98. If *abcd* . . . be an aspect of a whole system S of n units, so that all the aspects undistinguished from it are aspects of the whole system S, there being in all m undistinguished aspects, the different transpositions of sec. 97 will give us all the different systems of aspects of S referred to in sec. 96.

99. If the array of letters *abcd* . . . representing any aspect of a whole system S be given, and also those arrays representing all the other aspects of S which are undistinguished from *abcd* . . ., then the form of S is given; *i.e.*, if *pqrs* . . . be an aspect of any component collection of S, we know what other aspects of component collections of S are distinguished and what undistinguished from *pqrs* . . . For add letters to *pqrs* . . . until we get an array *pqrs* . . . *lmn* . . . representing an aspect of the whole system S. If this is not the same as one of the given arrays, transpose the letters of any one of the latter so that it becomes *pqrs* . . . *lmn* . . ., and make precisely similar transpositions in the case of each of the other given arrays. If now *wxyz* . . . $>\!\!\!-\!\!\!-\!\!\!<$ *pqrs* . . ., it follows from sec. 94 that there must be an aspect *wxyz* . . . *ijk* . . of the whole system S, such that *wxyz* . . . *ijk* . . . $>\!\!\!-\!\!\!-\!\!\!<$ *pqrs* . . . *lmn* . . . ; and if *wxyz* . . . $\longleftrightarrow$ *pqrs* . . . it follows from sec. 95 that there is no aspect *wxyz* . . . *ijk* . . . of the whole system S such that *wxyz* . . . *ijk* . . . $>\!\!\!-\!\!\!-\!\!\!<$ *pqrs* . . . *lmn* . . .; *i.e.*, in the former case there must be a symbol *wxyz* . . . *ijk* . . . among the transposed arrays, in the latter case there can be no such symbol.

Tabular Representation of Systems.

100. A convenient mode of arranging the symbols *abcd* . . ., &c., representing a system of m undistinguished aspects of a whole system S of n units a, b, c, d, . . . is

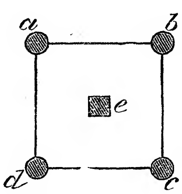


Fig. 23.

to place them one above another so that letters occupying "the same position" in the different rows may lie in the same column. For example, the system of fig. 23 may

be represented thus :—

$$\begin{array}{ccccc} a & b & c & d & e \\ b & c & d & a & e \\ c & d & a & b & e \\ d & a & b & c & e \\ a & d & c & b & e \\ d & c & b & a & e \\ c & b & a & d & e \\ b & a & d & c & e \end{array}$$

101. The symbol thus arrived at, consisting of nm letters arranged in m rows and n columns, will be termed the *tabular representation* of S. Each of the nm letters will be termed an *element.*

102. The order of the rows is clearly immaterial. Any alteration in the order of the columns merely substitutes for one system of undistinguished aspects of S another system of undistinguished aspects of S, and any one of these systems of undistinguished aspects defines the form of S, thus the order of the columns is also immaterial. The material point is that certain elements are all in the same row, and certain elements all in the same column.

103. Each sort of letter represents a unit of S; each column regarded as a unit represents a sort of place; each row represents an aspect of S, or, regarded as a unit, a unified aspect of S; and each letter, or element, an aspect of a unit of S.

104. Any row R_2 may be regarded as derived from another row R_1 by a substitution of the letters of the row R_1. If the same substitution be effected upon any row R_3, we get a row R_4 which is also a row of the table.

105. The substitutions by which any row of the tabular representation of S is derived from another may be said to be *substitutions proper to S.*

106. If we confine our attention to certain rows and columns only of the tabular representation of S, we get a table which will be termed a *constituent* of the whole table. The order of the rows and columns will be disregarded in a constituent as in a complete table.

107. If $a \longleftrightarrow b$, a and b can never appear in the same column. Thus the columns break up into lots, each lot belonging to a single system.

108. If $a >\!\!\!-\!\!\!< b$, then a appears in every column that b does, and b in every column that a does.

109. Generally if $abcd \ldots >\!\!\!-\!\!\!< pqrs \ldots$ we have in the table a constituent $\begin{matrix} a\ b\ c\ d \ldots \\ p\ q\ r\ s \ldots \end{matrix}$, and if $abcd \ldots \longleftrightarrow pqrs \ldots$ we have no such constituent. Thus the forms of the various components can readily be ascertained from the table.

110. If we desire to consider the forms of portions only of a system, we may

confine our attention to such constituents as will exhibit the form. Thus in the case of sec. 100 the constituent $\begin{smallmatrix} a & c \\ c & a \end{smallmatrix}$ shows that the pair a, c is a symmetrical one.

111. A discrete heap will be represented by a table of one row; a single heap of n units by a table of $|\underline{n}$ rows; and a heap consisting of single independent heaps of $p, q, r, s, \ldots$ units respectively will have a table of $|\underline{p} \times |\underline{q} \times |\underline{r} \times |\underline{s} \ldots$ rows.

Correspondences of Undistinguished Collections.

112. The various correspondences of undistinguished collections are indicated by two-row constituents of the tabular representation of the system of which the collections are components.

113. If the two rows of the constituent contain like letters in different orders, the constituent will be said to indicate a *self-correspondence, i.e.*, a correspondence of a collection C with itself; and not with another collection (sec. 76). Any two-row constituent containing two complete rows of the table representing a system S will indicate a self-correspondence of S, and the table may accordingly be said to indicate the form of S by indicating all its self-correspondences.

114. If the two rows of the constituent representing a correspondence of C with itself contain like letters in the same order, the constituent will be said to indicate an *identical-correspondence* of C. The identical-correspondence may be regarded as included among the self-correspondences of C. The self-correspondence of a single unit is an identical-correspondence.

115. In general the table representing a system S will have rows which are partially alike, indicating that in some of the self-correspondences of S some of the components are identically-correspondent. For example, in sec. 100 the unit e is always identically-correspondent. Thus constituents may have several rows which are duplicates. If we merely desire to consider those correspondences which the constituent indicates, we may, of course, omit duplicate rows.

116. If one two-row constituent be a part of another, the correspondence indicated by the former may be said to "occur in" that indicated by the latter; as when the latter correspondence occurs the former also occurs.

117. If we regard all collections of units under consideration in an investigation as components of the universal system which comprises all units, or, as is sufficient, as components of the whole system of units considered in the investigation, we may regard every correspondence of undistinguished collections as occurring in one or more self-correspondences of the universal or more limited system. From this point of view we see that a correspondence of two undistinguished collections, or the self-correspondence of a collection, restricts the possible correspondences and self-correspondences of other collections. In some cases it may determine absolutely the correspondences and self-correspondences of the other collections, not permitting

alternative correspondences or self-correspondences, in others it may partially restrict them, in others it may exercise no controlling effect whatever, so that collections may go through all correspondences and self-correspondences with as much freedom as if there were no correspondence already existing.

118. If two collections are such that each can go through all its self-correspondences while the other remains identically-correspondent, they will be said to be *independent*. If two collections are not independent, they will be said to be *related.*

119. If m and m' be the number of self-correspondences of two systems S and S′ respectively, the table representing the joint system S, S′ will have mm' rows if S and S′ are independent.

120. In the self-correspondences of a system every correspondence of undistinguished components occurs; but it is not of course in general the case that in the self-correspondences of any collection all the correspondences of undistinguished components of that collection occur. For example, in sec. 100 the only self-correspondence of a, b, c, is given by the constituent $\begin{smallmatrix} a & b & c \\ c & b & a \end{smallmatrix}$, and here no correspondence of the undistinguished units a, b occurs.

121. When investigating the correspondences of a number of $(n + m)$-ads we may for some time be occupied with the consideration of correspondences in which m of the units always remain identically-correspondent. The absence of change in the correspondence of the m-units may lead us to forget or overlook the fact that we are considering correspondences of the $(m + n)$-ads, and we may suppose that we are dealing with n-ads only. When then we find that a certain correspondence of two n-ads apparently does not exist, we must look closely to see whether we are not really considering correspondences of $(n + m)$-ads, and whether a change in the correspondence of the m units may not lead to the correspondence of the n which is supposed not to exist.

122. Units which in any correspondence of two undistinguished aspects are identically-correspondent may be termed the *foci* of the correspondence. Three or more undistinguished aspects such that the foci of the correspondences are the same in the case of each pair, may be said to be *confocal.* We may also use the term *confocal* as applicable to the case of corresponding components of the corresponding confocal aspects.

123. In cases in which, as mentioned in sec. 121, we consider $(n + m)$-ads as if they were n-ads, we really pass from the consideration of the original units to that of other units which are arrived at by taking the m foci with each of the other units successively, and then regarding the resulting $(m + 1)$-ads as single units.

124. Differences between things can always be ignored, and thus we may at one time regard two collections of units as distinguished, and at another, by ignoring differences between them, as undistinguished. Here we are dealing with certain units, upon which the differences depend, in addition to those of the two collections.

125. For example in the case of the system of fig. 24, although all the graphical units are alike if taken apart from the barbed lines joining them, the two triads

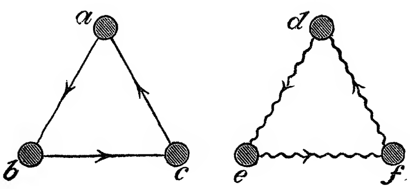


Fig. 24.

a, b, c and d, e, f are distinguished because plain lines are distinguished from wavy ones. If we ignore the difference between straightness L and waviness W, we are really dealing with the eight units a, b, c, d, e, f, L, W, and instead of dealing with units a, b, c, &c., are really dealing with the triads aLW, bLW, &c. When we regard the two triads a, b, c and d, e, f as distinguished, we deal with correspondences represented by the constituent consisting of the first, or last, nine rows of the table at the end of the section, in which it will be seen that L and W remain identically-correspondent, and the correspondences are such as would exist if L and W were distinguished. When we ignore the difference between straightness and waviness, we admit correspondences such as $\begin{matrix} a\,b\,c\,d\,e\,f\,\mathrm{L}\,\mathrm{W} \\ d\,e\,f\,a\,b\,c\,\mathrm{W}\,\mathrm{L} \end{matrix}$.

The table representing the whole system of eight units is as follows:—

a	b	c	d	e	f	L	W
a	b	c	e	f	d	L	W
a	b	c	f	d	e	L	W
b	c	a	d	e	f	L	W
b	c	a	e	f	d	L	W
b	c	a	f	d	e	L	W
c	a	b	d	e	f	L	W
c	a	b	e	f	d	L	W
c	a	b	f	d	e	L	W
d	e	f	a	b	c	W	L
d	e	f	b	c	a	W	L
d	e	f	c	a	b	W	L
e	f	d	a	b	c	W	L
e	f	d	b	c	a	W	L
e	f	d	c	a	b	W	L
f	d	e	a	b	c	W	L
f	d	e	b	c	a	W	L
f	d	e	c	a	b	W	L

126. Many of the relations which collections of units hold to each other are apparently independent of the form of the collections, or of the form of the system S, of which they are components. In dealing with such relations we ignore differences, and regard the units composing the collections as single heaps, *i.e.*, we do not deal with the units $a, b, c, d, \ldots$ of S, but with others $\alpha, \beta, \gamma, \delta, \ldots$ which constitute a single heap, but have the same self-correspondences as $a, b, c, d, \ldots$ have as long as certain other units remain identically-correspondent.

127. Since we can in all cases ignore differences, any system S may be regarded as a single heap, the peculiarities of form which it possesses in any particular investigation being regarded as due to the fact that we are not considering S alone, but in conjunction with other units, those correspondences only of S being dealt with which admit of the additional units remaining identically-correspondent. All statements, therefore, as to the distinguishableness and undistinguishableness of components of S, and as to their being of particular forms may be taken as relative, viz., as statements that the components have the correspondences characteristic of those forms as long as certain units detached from S remain identically-correspondent.

128. It may be laid down generally, that in almost every instance where we seem to investigate a base system S which may be regarded and spoken of as being of n units and of a particular form, we really deal with a single heap system H of n units and a system F which remains identically-correspondent while the units of H go through the correspondences characteristic of the system S.

129. Any collection of units which while another collection C remains identically-correspondent has self-correspondences characteristic of a collection of a special form may be said to be of that form relatively to C.

Sets.

130. If $abcd \ldots$, $pqrs \ldots$, are undistinguished components of a collection $a, b, c, d, \ldots p, q, r, s, \ldots l, m, n, o, \ldots$, then the units $w, x, y, z, \ldots$ which are such that $abcd \ldots lmno \ldots \rightarrowtail\!\!\!-\!\!\!\leftarrowtail pqrs \ldots wxyz \ldots$ may or may not be units of the collection, and in some cases cannot be selected so as to be units of the collection. If the collection be such that whatever undistinguished components $abcd \ldots$, $pqrs \ldots$ we select, and whatever other component $lmno \ldots$ we select, $w, x, y, z \ldots$ can always be selected from the collection, then the collection will be termed a *set*.

131. A system is obviously a set. A set is not necessarily a system; it may be one of a number of undistinguished sets which together compose a single or multiple system. In most investigations our inquiries are directed towards the discovery of the forms of component sets of the base system. As far as the distribution of its distinguished and undistinguished components is concerned, a set in no way differs from a system.

132. In the self-correspondences of a set every correspondence of its undistinguished

components occurs. Thus the constituent which represents a set has sub-constituents indicating all the correspondences of the components of the set.

133. In a graphical representation of a set the diagram not only indicates what units, &c., are distinguished and what undistinguished, but is such that if we ignore geometrical position in accordance with sec. 41, and the reference letters in accordance with sec. 42, the units, &c., will be actually distinguished and undistinguished, and the diagram will represent the form of the set by being actually of that form.

134. Any collection of units which is not a set may be said to be *imperfect.*

135. If any collection of units is such that its component pairs are all distinguished from the pairs connecting it with units detached from it, the collection is a set. For let *lmn* . . . and *pqr* . . . be any two undistinguished components of the collection, then if *a* be any unit of the collection, a unit *b* which is such that *lmn* . . . *a* is undistinguished from *pqr* . . . *b* must also be a component of the collection, otherwise the pairs connecting *b* and *p, q, r,* . . . will be pairs connecting a unit not of the collection with units of the collection, and they will be undistinguished from the pairs connecting *a* and *l, m, n,* . . . which are all pairs of the collection.

136. Let *x* be any unit of a single set Q of *n* units; consider the pairs formed by *x* with other units of Q; take any one of these *xp;* let the number of pairs *xp, xq, xr,* . . . which are undistinguished from *xp,* be *m;* then in the case of any other unit *y* of Q the number of pairs *yi, yj, yk,* . . . which are undistinguished from *xp* is *m* also. Further the number of pairs *ax, bx, cx,* . . . which are undistinguished from *xp* is also *m;* for the number *m′* of such pairs must be the same in the case of each unit, and thus we have $mn=m'n$, i.e., $m=m'$.

Aspects unique with respect to Collections.

137. If *xyz* . . . *abc* . . . $>\!\!-\!\!-\!\!<$ *uvw* . . . *abc* . . ., then the aspects *xyz* . . . and *uvw* may be said to be *duplicates with respect to* the collection *a, b, c,* . . . If there is no duplicate of *xyz* . . . with respect to *a, b, c,* . . . then *xyz* . . . may be said to be *unique with respect to a, b, c,* . . . It should be observed that if *xyz* . . . is unique with respect to *a, b, c,* . . . there may be an aspect *uvw* . . . such that *xyz* . . . *abc* . . . $>\!\!-\!\!-\!\!<$ *uvw* . . . *cab* . . .

138. If the aspect *abc* . . . be unique with respect to the collection *d, e, f,* . . . and if the apsect *def* . . . be unique with respect to the collection *g, h, i,* . . . ; then *abc* . . . is unique with respect to *g, h, i,* . . . I give the proof in the case in which the three collections are detached; the proof in other cases is somewhat longer but presents no difficulty. If *abc* . . . is not unique with respect to *g, h, i,* . . . there are units *p, q, r,* . . . such that *abc* . . . *ghi* $>\!\!-\!\!-\!\!<$ *pqr* . . . *ghi* . . ., and there are (sec. 94) units *s, t, u,* . . . such that *abc* . . . *def* . . . *ghi* . . . $>\!\!-\!\!-\!\!<$ *pqr* . . . *stu* . . . *ghi* . . . ; but *def* . . . is unique with respect to *g, h, i,* . . . therefore *stu* . . . is *def* . . . , therefore *abc* . . . *def*

. . . $>\!\!\!-\!\!\!-\!\!\!<$ *pqr* . . . *def* . . . , therefore *abc* . . . is not unique with respect to *d*, *e*, *f*, . . . contrary to the hypothesis; *i.e.*, *abc* . . . is unique with respect to *g*, *h*, *i*, . . .

139. If a single unit x be unique with respect to a collection *a*, *b*, *c*, . . . we may represent x by the symbol (*abc* . . .) or [*abc* . . .] &c. The two symbols (*abc* . . .) and [*abc* . . .] will, where the different sorts of brackets are distinguished from each other, represent two units which are distinguished from each other, and each unique with respect to *a*, *b*, *c*, . . . ; or we may use the two symbols to represent two undistinguished units *p*, *q* which are each unique with respect to *a*, *b*, *c* . . . , the difference between the brackets indicating that the units *p*, *q* are not identical.

140. If *abc* . . . $>\!\!\!-\!\!\!-\!\!\!<$ *pqr* . . . then (*abc* . . .) $>\!\!\!-\!\!\!-\!\!\!<$ (*pqr* . . .), where (*abc* . . .) and (*pqr* . . .) may be the same or different units; and (*abc* . . .) *abc* . . . $>\!\!\!-\!\!\!-\!\!\!<$ (*pqr* . . .) *pqr* . . . If *abc* . . . $\longleftrightarrow$ *pqr* . . . then (*abc* . . .) and (*pqr* . . .) may or may not be distinguished units.

141. If we consider aspects such as P*abc* . . ., Q*abc* . . ., P*def* . . ., &c., where in each aspect one only of the units P, Q, &c., appears, we may represent P, Q, &c., by brackets of different sorts, and write the units which are unique with respect to P, *a*, *b*, *c*, . . ., &c., thus (*abc* . . .), [*abc* . . .], (*def* . . .), &c., where P is represented by the brackets ( ), Q by the brackets [ ].

142. Suppose $a \equiv (bc)$, $b \equiv (de)$, $e \equiv (fc)$, then we may represent a by the symbol $((de)\,c)$ or $((d\,(fc))\,c)$, each symbol representing a and at the same time an aspect of a collection of which a is a component. We may have such symbols with various different sorts of brackets, *e.g.*, if $d \equiv [mn]$, a will be represented by $(([mn]\,e)\,c)$.

Associates.

143. If *a*, *b*, *c*, . . . be any collection of units, and if λ be another unit, such that the pairs λa, λb, λc, . . . are distinguished from all pairs which λ makes with units which are not components of the collection, λ may be said to be an *associate* of the collection *a*, *b*, *c*, . . .

144. If the pairs λa, λb, λc, . . . are all undistinguished from each other, λ may be termed a *single pair associate.*

145. If some of the pairs are distinguished from each other, λ may be termed a *multiple pair associate.*

146. In the special case in which all the pairs are distinguished from each other a multiple pair associate may be termed a *discrete pair associate.*

147. A single pair associate of a collection can only exist if all the units of the collection are undistinguished from each other. Also in the case of a multiple pair associate of a collection units of the collection which make pairs with the associate which are undistinguished must themselves be undistinguished.

148. An associate of any description may be graphically represented by a graphical unit connected by links with each of the units of which it is an associate.

149. If we desire in the case of a multiple pair associate to indicate the distinguished pairs, we may employ lines of various sorts in lieu of links, thus :—

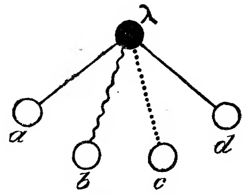


Fig. 25.

150. Many of our conceptions and definitions of systems of units involve the idea of associates, and the graphical definition of systems by means of links is very convenient as enabling this to be visually represented.

151. In many cases units which we might at first sight regard as associates of a system S of some definite form, are really associates of a single heap system H accompanied by a system F which remains self-correspondent while H goes through correspondences characteristic of the form of S (sec. 128).

Unified Aspects.

152. The unit arrived at by regarding any aspect of a collection $a, b, c, \ldots$ as a single unit, is unique with respect to the collection and may be represented by the symbol $(abc \ldots)$.

153. Two aspects when unified are two distinct units, and not one unit. Thus if $abc \ldots$ and $lmn \ldots$ are different aspects $(abc \ldots)$ is not $(lmn \ldots)$.

154. Every aspect is unique with respect to its unified aspect; for if $abc \ldots (abc \ldots) >\!\!-\!\!-\!\!< lmn \ldots (abc \ldots)$, then $abc \ldots >\!\!-\!\!-\!\!< lmn \ldots$ and then (sec. 140) $abc \ldots (abc \ldots) >\!\!-\!\!-\!\!< lmn \ldots (lmn \ldots)$; thus we have $lmn \ldots (abc \ldots) >\!\!-\!\!-\!\!< lmn \ldots (lmn \ldots)$; and $(lmn \ldots)$ is not $(abc \ldots)$ (sec. 153), thus $(lmn \ldots)$ is not unique with respect to $lmn \ldots$, which is contrary to sec. 152; therefore, &c.

155. If λ and μ be two unified aspects of the same collection they are unique with respect to each other. For, by sec. 152, λ is unique with respect to the collection, and, by sec. 154, the aspect of the collection which is μ when unified is unique with respect to μ, thus, by sec. 138, λ is unique with respect to μ. Similarly μ is unique with respect to λ.

156. The pairs which a unified aspect $(abc \ldots)$ makes with $a, b, c, \ldots$ respectively are all distinguished from each other. For if $(abc \ldots)a >\!\!-\!\!-\!\!< (abc \ldots)b$, then there are units $l, m, \ldots$ such that $(abc \ldots)abc \ldots >\!\!-\!\!-\!\!< (abc \ldots)blm \ldots$, *i.e.*, such that $abc \ldots >\!\!-\!\!-\!\!< blm \ldots$, *i.e.*, such that $(abc \ldots)abc \ldots >\!\!-\!\!-\!\!< (blm \ldots)blm \ldots$, *i.e.*, such that $(abc \ldots)blm \ldots >\!\!-\!\!-\!\!< (blm \ldots)blm$, *i.e.*, such that $(blm \ldots)$ is not unique with respect to $blm \ldots$, contrary to sec. 152, therefore, &c.

157. In the same way it may be shown that the pairs which $(abc\ldots)$ makes with $a, b, c, \ldots$ respectively are all distinguished from the pairs which $(abc\ldots)$ makes with any other units $x, y, z, \ldots$ not components of the collection $a, b, c, \ldots$

158. A unified aspect $(abc\ldots)$ is accordingly a discrete pair associate of the collection $a, b, c, \ldots$ (sec. 146).

159. It should be observed that in the case of a number of undistinguished aspects no unit can be said to be *the* unit arrived at by regarding any particular one of those aspects as unified. All we can say is that any one of a number of units may be so regarded. If, however, one of that number is regarded as representing a particular one of the unified aspects, some other definite unit of the number must be regarded as the unit which represents another given unified aspect, *i.e.*, the two unified aspects will be such that they can only be represented by certain pairs of the units and not by any pair.

160. When we represent aspects by single letters, those letters really represent the unified aspects. Whatever relations as to distinguishableness or undistinguishableness exist between the aspects, there will be precisely the same relations between the unified aspects; so that we may deal with either the unified or non-unified aspects. Thus if A, B, C, D, be the aspects *abcd*, *efgh*, *ijkl*, *uvwx*, when regarded as units, then if $abcdefgh >\!\!-\!\!\!-\!\!< ijkluvwx$, we have $\mathrm{AB} >\!\!-\!\!\!-\!\!< \mathrm{CD}$.

161. The conception of a unified aspect is an "accidental" one; for the units representing unified aspects of components of a system S may be regarded as representing any other things holding similar relations to the units of S. The method of defining systems by regarding their units as unified aspects of components of other systems is, however, so convenient and simple that it will be frequently employed, and the accidental part of the definition being borne in mind, no danger can arise from the employment of this method of arriving at systems.

Correspondences of Systems of like Forms.

162. We may consider correspondences of any two independent systems S_1 and S_2 of the same form, in which every component of S_1 corresponds to a component of S_2 of the same form, and we may regard these correspondences as units. The number of such correspondences is in general greater than the number of self-correspondences of each system, as in the case of the latter we only suppose undistinguished components to correspond, and do not admit correspondences of distinguished components of like form.

163. If, however, we start with a correspondence of S_1 and S_2 and then restrict ourselves as regards other correspondences to those in which each component of S_1 corresponds only with such components of S_2 as are undistinguished from that with which it corresponded in the first instance, we shall get the same number of correspondences as there are self-correspondences of each collection S_1, S_2.

164. If we then take another correspondence not included among the former, we shall get a second collection of correspondences equal in number to the former, and so on until the correspondences are all exhausted.

165. For example the six graphical units of the system of fig. 26 constitute two

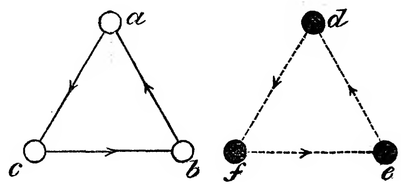


Fig. 26.

independent systems of the same form. Starting with the correspondence $\frac{abc}{def}$ which we may represent as a unit P, we may add correspondences $\frac{abc}{efd}$ and $\frac{abc}{fde}$ which we may represent by units Q and R respectively. Here we have only correspondences such as those referred to in sec. 163. It is to be observed that we cannot properly say that P represents any particular one of the three correspondences, for they are undistinguished; the three units P, Q, R, together represent the three correspondences; but if P is regarded as representing any definite correspondence Q and R will each represent definite correspondences, for P, Q, R, are each unique with respect to each other. If we now consider the correspondence $\frac{abc}{dfe}$ in which we have components of the same form corresponding as before, we get the correspondences $\frac{abc}{fed}$ and $\frac{abc}{edf}$. We may represent these by the units L, M, N. The whole system of units considered is represented by the following table—

a	b	c	d	e	f	P	Q	R	L	M	N
a	b	c	e	f	d	Q	R	P	N	L	M
a	b	c	f	d	e	R	P	Q	M	N	L
b	c	a	d	e	f	R	P	Q	N	L	M
b	c	a	e	f	d	P	Q	R	M	N	L
b	c	a	f	d	e	Q	R	P	L	M	N
c	a	b	d	e	f	Q	R	P	M	N	L
c	a	b	e	f	d	R	P	Q	L	M	N
c	a	b	f	d	e	P	Q	R	N	L	M

166. We may of course consider self-correspondences of a system S in which components of like form correspond which are distinguished from each other. Thus in the case of the system a, b, c, d, e, of fig. 27 we may consider correspondences such

as $\frac{abcde}{aedcb}$ or $\frac{abcde}{acebd}$. It is, however, to be understood that the term "self-correspondence" will always be used, as heretofore, with reference to self-correspondences which involve the correspondence of undistinguished collections only.

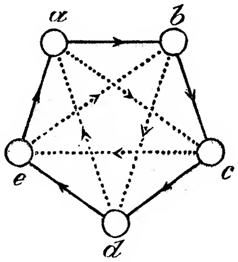


Fig. 27.

167. The two systems S and X of sec. 81 are of the same form. In the tabular representation of either the unified columns will represent the units of the other. Further, S and X are independent systems, no special relation except that of similarity of form exists between them. Each aspect of S is arrived at by considering a correspondence of S and X; it may in fact be regarded as such a correspondence, so that a unified aspect is a unified correspondence, and the whole system of unified aspects of S represent a system of unified correspondences of two systems of the same form, in which we are restricted to such correspondences as those referred to in sec. 163.

168. When two systems are regarded as corresponding they may be spoken of as being *projections* of each other in as many ways as there are unified correspondences.

169. If we represent the units of one of two independent systems of the same form by the symbols (λa), (λb), (λc), &c., we may represent those of the other by the symbols (μa), (μb), (μc), &c., where in one correspondence we may conveniently suppose that (λa) corresponds to (μa), (λb) to (μb), and so on; but it is not to be supposed that this correspondence is distinguished from others.

Replicas.

170. If $a, b, c, d, \ldots$ and $\alpha, \beta, \gamma, \delta, \ldots$ be two systems of units such that a and α are unique with respect to each other, as also b and β, c and γ, &c., and if, a and b being any two units of the first system, when $a \succ\!\!-\!\!\prec b$ we have also $a\alpha \succ\!\!-\!\!\prec b\beta$, then $\alpha, \beta, \gamma, \delta, \ldots$ may be called a *replica* of $a, b, c, d, \ldots$

171. The replica of a system S is of the same form as S, and has the same relations to other systems as S has. In the tabular representation of S and its replica, whatever transpositions of the letters representing an aspect of S takes place in passing from one row to another, precisely the same transposition takes place in the case of the letters representing an aspect of the replica.

172. The replica of a multiple system S may have some single systems in common with S; some of the single systems may be their own replicas.

173. The letters placed adjacent to graphical units for the purpose of reference compose a system which is a replica of the graphical system.

Independent and Related Systems.

174. Let S, S′ be a system consisting of two detached systems S and S′; let $a, b, c, d, \ldots$ be the units of S, n the number of those units, and m the number of the self-correspondences of S. Let $a', b', c', d', \ldots$ be the units of S′, n' the number of those units, and m' the number of the self-correspondences of S′. Let μ be the number of self-correspondences of the system S, S′: μ cannot be $> mm'$, otherwise there would be duplicate rows in the tabular representation of S, S′. Consider now a set of rows in the tabular representation of S, S′ obtained by taking any row R and all others in which the letters $a', b', c', d', \ldots$ remain untransposed from the order they have in R. Let the number of rows in the set be k. A transposition indicated by two rows of the set if made to operate on any row of the set will clearly give a row also of the set. Now take any fresh row not of the set; this must also be one of a set of k rows detached from the former set. Thus all the rows divide up into m' sets of k rows, and we have $m'k=\mu$. If k' be the corresponding number in the case where $a, b, c, d, \ldots$ remain untransposed, we have $mk'=\mu$. Thus we have $m'k=mk'=\mu=mm'$ or $< mm'$.

175. Let $\mu=mm'$, then $k=m$, $k'=m'$, and the two systems are independent.

176. We must have $mm'=\mu$ unless m and m' have a common integral factor; so that two systems S and S′ must be independent unless the numbers of their respective self-correspondences have a common factor. The two systems may of course be independent when m and m' have a common factor.

177. If m and m' are prime to each other S and S′ may be said to be *prime* to each other; so that systems which are prime to each other are independent.

178. If $mm' < \mu$, m and m' must have a common factor, and the two systems S and S′ will be *related*. If $l'm'n' \ldots$, $x'y'z' \ldots$ are undistinguished components of S′, and $abcd \ldots$ be an aspect of the whole system S, then the aspects $l'm'n' \ldots abcd \ldots$ and $x'y'z' \ldots abcd \ldots$ will not in general be undistinguished from each other; and any graphical diagram representing the system S, S′ must have lines, or successions of lines, connecting units of S to units of S′, so that S and S′ cannot be drawn on separate sheets of paper.

179. If we have three detached systems, S_1, S_2, S_3; and S_1 and S_2 are related, and also S_1 and S_3, it does not follow that S_2 and S_3 are related; the single consideration that m_1 and m_2 may have a common factor, and also m_1 and m_3, without m_2 and m_3 having one, shows this.

180. If $m = m' = \mu$, then S and S′ are replicas of each other.

181. If $m = \mu$, and $m \neq m'$, m' will be a factor of m, and S′ may be said to be a *factor system* of S. We have here $m = m'k = \mu$.

182. If S′ be a factor system of S we have $k' = 1$; *i.e.*, there is no self-correspondence of S, S′, other than the identical-correspondence, in which S is identically-correspondent. If S′ is not a factor system of S, we have $k > 1$, and there are self-correspondences of S, S′, other than the identical-correspondence, in which S is identically-correspondent.

183. Hence if S′ be a factor system of S, there are no units of S′ which are duplicates with respect to S. And if there are units of S′ which are duplicates with respect to S, then S′ is not a factor system of S. Further, if S′ be not a factor system of S, there are units of S′ which are duplicates with respect to S, unless S and S′ are replicas of each other. And if there are no units of S′ which are duplicates with respect to S, then S′ is a factor system of S, unless S and S′ are replicas of each other.

184. If S′ be a factor system of S, each single system of S′ must be so also. For if m_1' be the number of self-correspondences of S_1' a single system of S′, we have $m_1' \not> m'$, *i.e.*, $m_1'm \not> mm' < \mu$, *i.e.*, $m_1'm < \mu$.

185. If S′ be a factor system of S, it is also a factor system of Δ, the single system composed of the unified aspects of S. This follows immediately from the fact that the combined system S, Δ has the same number of self-correspondences as S has.

186. Hence also every single system of S′, a factor system of S, must be a factor system of Δ.

187. A system which has no factor systems, except systems of one unit, may be termed a *prime system*.

188. If S″ is a factor system of S′, and S′ is a factor system of S, then S″ is a factor system of S. For let a'', b'' units of S″ be duplicates with respect to S; then we have $a''\mathrm{S} \rightarrowtail\!\!\!\leftarrowtail b''\mathrm{S}$. Now if c' be any unit of S′, there must be a unit d' such that $a''c'\mathrm{S} \rightarrowtail\!\!\!\leftarrowtail b''d'\mathrm{S}$; and since S′ is a factor of S, we cannot have $c'\mathrm{S} \rightarrowtail\!\!\!\leftarrowtail d'\mathrm{S}$, unless c' is d'. Thus we have $a''c'\mathrm{S} \rightarrowtail\!\!\!\leftarrowtail b''c'\mathrm{S}$, *i.e.*, we have $a''c' \rightarrowtail\!\!\!\leftarrowtail b''c'$, whatever unit c' of S we take, *i.e.*, a'' is b'', and there are no units of S″ duplicates with respect to S, *i.e.*, S″ is a factor system of S.

Three Modes of Compounding Systems.

189. There are three modes of deriving a system from two or more other systems to which a passing reference may here be made. In the first the compound system is arrived at by regarding the n-ads connecting n independent or related systems as units.

190. Again, we may have a system which may be regarded as composed of n independent (and therefore detached) undistinguished sets, each of a given form F, and such that their unified aspects compose a system of the form F′.

191. An important special case of such a compound system is that in which the form

F is that of a single heap H of m units. Here F′ being the form of any system S, the compound system S_H differs from S in that we have in lieu of any unit a of S a single heap of m units, each unit of which may be called a, the form relations between the connecting r-ads of r of the heaps being the same as those existing between the r-ads of S. Thus if in S we have $abcd \succ\!\!-\!\!\prec pqrs$, in S_H we shall also have $abcd \succ\!\!-\!\!\prec pqrs$, whatever units of the heaps $aaa \ldots$, $bbb \ldots$, &c., we select.

192. If we combine two connected component collections L and M of S, the resulting collection cannot contain those units which are common to L and M twice over; *i.e.*, the result of combining the collections a, b, c, d, and c, d, e, f is the collection a, b, c, d, e, f. If in S_H we select collections a, b, c, d and c, d, e, f so that the units c, d of the first are not the same units as c, d of the second, the sum of the two collections will be the collection a, b, c, c, d, d, e, f.

193. The third mode of composition is that in which the derived system has all the self-correspondences which each of a number of systems, of the same number of units, has. We may represent the units of such a system by symbols

$$(\mathrm{A}a \ldots),\ (\mathrm{B}b \ldots),\ (\mathrm{C}c \ldots),\ \&c.,$$

where A, B, C . . . are units of one of the compounded systems, a, b, c those of another, and so on.

General Method of Graphically Representing a System.

194. Let each unit of a system S, each unified column, each unified row, and each element of its tabular representation, be represented by a graphical unit, using different kinds of graphical units in the case of the units of S, the rows, columns, and elements respectively, four kinds in all. Connect each graphical unit which represents an element by links to

(1.) the graphical unit representing the unit of S of which the element represents a unit aspect;

(2.) the graphical unit representing the unified row in which the element lies;

(3.) the graphical unit representing the unified column in which the element lies.

We get a graphical representation of a system of $2n+m+mn$ units, of which S is a component system.

195. It is obviously not always necessary to employ the somewhat cumbrous mode of graphical representation here given; simpler methods can be adopted in special cases. Frequently, as we have already seen, it will not be necessary to have any auxiliary graphical units, but merely those representing the units of S itself.

Networks.

196. The pairs of any single collection of pairs are either all component pairs of a single system of units, or all connecting pairs of two single systems of units.

197. Where a single system P of pairs connects two single systems S and Σ, the number of pairs of P which connect any unit a of S to units of Σ is the same as the number of pairs of P which connect any other unit b of S to units of Σ. The number in the case of units of S is the same as the corresponding number in the case of units of Σ if S and Σ contain the same number of units, but not otherwise.

198. Where the pairs of a single system of pairs are components of a single system S of units, the number μ of pairs of the former system which connect any unit a of S to other units of S is the same as that in the case of any other unit b of S. Also the number of such pairs connecting units of S to a is the same as that of those connecting units of S to any other unit b of S, being in both cases also μ (sec. 136).

199. A single system of component pairs of any single system of units constitutes a *simple network* of which the number μ of the last section may be called the *way-number.* This network consists of one or more portions each continuous and detached from the other portions. Where there are two or more detached continuous portions of a simple network, each is undistinguished from the others, for component pairs of distinguished portions would be distinguished.

200. Every simple network is accompanied by one in which the pairs connect the same units but in the reverse order, so that if ab is a pair of one network, ba is a pair of the other. The two networks may be called the *reverses* of each other. If we represent the unified pairs of the one by single letters $\alpha, \alpha, \alpha, \alpha$, we may represent those of the other by $\alpha', \alpha', \alpha', \alpha'$. Here the sorts of the letters may be regarded as representing the networks when regarded as units.

201. We may use the symbol (α) to represent the network, of which α is a unified pair. Here (α) takes the place of the sort of α. Or we may use (ab) (sec. 263).

202. The units of a continuous detached portion of a simple network form a set, for any two are connected by a chain of one or more pairs of the network, while no component unit is connected by such a chain with a detached unit, so that all component pairs are distinguished from all connecting pairs of the collection (sec. 135).

203. Thus a simple network divides the units of a single system into one or more detached sets, each of the same number of units. If, then, ρ be the number of units in each set, and σ the number of sets, $\rho\sigma=n$ the number of units in the system. Thus ρ and σ must each be integral factors of n.

204. Equations such as

$$(\mathrm{S})=\frac{p}{a}\cdot\frac{q}{b}\cdot\frac{r}{c}\cdot\frac{s}{d},$$

where $pa=qb=rc=sd=n$, may be employed to denote the fact that there is a simple network which divides the units of S into p sets of a units, another which divides them into q sets of b units, and so on. We may have equations such as

$$(\mathrm{S})=\frac{p}{a}\cdot\frac{p}{a}\cdot\frac{p}{a}\cdot\frac{q}{b}\cdot\frac{q}{b},$$

showing that there are three different simple networks which divide S into p sets of a units, and so on. We may in such a case write the equation thus :—

$$(\mathrm{S})=\left(\frac{p}{a}\right)^3\cdot\left(\frac{q}{b}\right)^2$$

205. Two distinguished simple networks, components of a single system S, either divide S into the same detached sets, or the pairs of one of the simple networks are pairs connecting the detached continuous portions of the other. For pairs belonging to one simple network must either be all component pairs of the sets into which S is divided by the other simple network or all pairs connecting them.

206. A network composed of two or more simple networks may be called a *compound* network, and may be called *double, treble,* &c., according to the number of the component simple networks. A compound network will consist of one or more undistinguished detached continuous compound portions. The units of each of these portions constitute a set.

207. Where a network simple or compound consists of only one detached portion, it may be said to be *complete.*

208. If the pairs of each simple network of a compound network N connect detached portions of the compound network composed of the remaining simple networks of N, then the compound network N may be said to be *pure.*

209. Any two simple networks of a pure compound network may be said to be *outside* each other.

210. If each pair of a simple network connects units which are both components of the same detached portion of a simple or compound network, the latter network may be said to *enclose* the former.

Chains.

211. A succession of undistinguished pairs, ab, bc, cd . . . may be termed a *simple chain.* Where the chain has no terminal units it may be said to be *closed.* Every simple chain is a portion of a closed chain, unless it contains one pair only, when it may be a pair connecting two systems.

212. *Compound chains* are such as contain distinguished pairs, or undistinguished pairs of opposite polarities.

213. In the case of the pairs connecting two systems we may have networks consisting entirely of undistinguished pairs ; but every chain in each consists of successions of pairs alternately of opposite polarities.

214. A complete network of a system contains simple or compound chains connecting every two units of the system.

215. If a chain of pairs of a pure compound network contains one pair only from one of the component simple networks, it cannot be closed, for if it were the single

pair would not connect two detached portions of the compound network composed of the simple networks from which the remaining pairs are taken, but two units of a continuous portion.

216. The number of pairs in a closed chain may be termed its *period.* The period of a closed chain is of course the same as the number of units it connects.

217. A symmetrical pair of units may be regarded as constituting a simple closed chain of period 2; thus

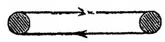


Fig. 28.

218. It may also be convenient in some cases to regard each unit as connected to itself by a single barbed line constituting a simple closed chain of period 1, thus

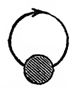


Fig. 29.

219. We may represent a chain of pairs thus $a\lambda b\mu c\nu d\rho e$, where a, b, c, d, e represent units, and the sorts of λ, μ, ν, ρ the networks of which the pairs ab, bc, cd, de, are respectively components; λ, μ, ν, &c., being the letters placed alongside of the plain lines if we employ the graphical method of sec. 64. In some cases it will not be necessary to employ specific letters to represent the units, they may be represented by the spaces between the letters representing the unified pairs, *e.g.*, we may represent the chain given above thus $a\lambda\mu\nu\rho e$, where only the terminal units are given. If we have $a\lambda\mu\nu\rho e$ and $a\pi\sigma e$, we see that the two units a, e are connected by the two different chains $\lambda\mu\nu\rho$ and $\pi\sigma$.

220. We may have equations such as

$$\lambda\mu\nu\rho=\pi\sigma=\kappa,$$

denoting the fact that the three chains $\lambda\mu\nu\rho$, $\pi\sigma$, κ can have the same terminals, and we may write a chain such as $\lambda\mu\mu\mu\nu$ for shortness $\lambda\mu^3\nu$, and similarly in other cases.

221. We may employ as the symbol to be attached to the barbed line of sec. 218 in accordance with the method of sec. 64 the symbol 1. An equation such as $\lambda\mu\nu=1$ will then show that the chain $\lambda\mu\nu$ on the left hand side of the equation is closed.

Groups—Circuits.

222. Any single set which is such that each component unit is unique with respect to each of the others, may be termed a *group.*

223. The single system Δ of unified aspects of any system (sec. 185) is a group, as each unified aspect is unique with respect to each of the others.

224. Every component set of a group is a group.

225. The table, or constituent of a table, representing a group of n units has n rows, a letter of any one sort appearing once and once only in each column.

226. The simple networks of a group are all one-way (sec. 199) networks. The closed chains which constitute the detached portions of any network of a group may be called *simple circuits.* Any closed compound chain of a group may be called a *compound circuit.*

227. If the component pairs of one simple circuit are undistinguished from those of another, the two circuits must have the same period.

228. If the period of a simple circuit be rs where r and s are integers, there will be r simple circuits of period s, and s of period r, connecting the units of the circuit.

229. Now the number of units in a simple circuit of a group of n units must be a factor of n (sec. 203); so that if n be a prime number, there is only one form of group; for one simple circuit must contain all the units, and this being given all the other circuits are given; they are all of period n.

230. In a group if the pairs of one circuit are distinguished from those of another, but both circuits are of the same period, they may be said to be *similar.* The pairs of such circuits may also be said to be similar.

231. If $a, b, c, d, e, \ldots$, A, B, C, D, E, ... be units of a group, and if $a, b, c, d, e, \ldots$ constitute a simple circuit in the order given, and if the pairs aA, bB, cC, dD, ... be all undistinguished from each other, then the units A, B, C, D, E, ... constitute a similar circuit in the order given.

232. In the case of a group the terminals of any chain $\lambda\mu\nu\rho$ constitute a pair of a definite simple network (α), *i.e.*, we cannot have $\lambda\mu\nu\rho=\alpha$, and $\lambda\mu\nu\rho=\beta$.

233. We may use the symbol $(\alpha\beta\gamma)$ to represent the network (λ) where $\lambda=\alpha\beta\gamma$.

234. The network of which the two terminals of a chain of pairs are a component pair, may be termed the *product* of the networks of which the pairs composing the chain are components; the order of the networks in the product being of course the same as the order of their component pairs in the chain.

235. A pure complete network furnishes chains connecting every pair of units of a group, and thus if $\alpha, \beta, \gamma, \delta, \ldots$ be unified pairs of the simple networks of such a network, we can with them make chains $\alpha\beta$, $\beta\gamma^2$, $\delta\alpha$, &c., whose terminals constitute pairs of every simple network of the group. A group is accordingly fully defined if one of its pure complete networks is given.

236. We have, whatever pair α may be, $\alpha\alpha'=1$ (sec. 200), and if α be a symmetrical pair $\alpha=\alpha'$, and thus $\alpha^2=1$.

237. If $\alpha\beta=\beta\alpha$, the two networks (α) and (β) may be said to be *commutative.*

238. If a simple network is commutative with each of a number of others, it is commutative with every simple network which is enclosed in the compound network composed of the latter.

239. The system S arrived at by compounding together a number of independent groups in the mode described in sec. 189 is a group having component sets of the form

of each of the original groups. We may fully define the group S by a pure complete network composed of a number of pure networks such that each detached portion of one is composed of units constituting a group of the form of one of the original groups.

Groups containing from one to twelve Units.

240. In the following twelve sections I shall denote the number of units in a group by n; f will denote the number of forms of groups for any value of n; G the graphical representation; T the tabular; and (S) will be the symbol described in section 204. Where f is greater than unity, the symbols G, T, and (S) will have numerical suffixes corresponding to the different forms. In the graphical representation, only such lines will be drawn as are necessary to completely define the group, and in some cases alternative representations will be given, obtained by taking different circuits. The values of n taken are from 1 to 12 inclusive.

241. If $n=1$ we have $f=1$,

$(S)=\frac{1}{1}$, G= 

, $T=a$.

Fig. 30.

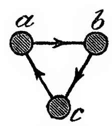
242. If $n=2$ we have $f=1$,

$(S)=\frac{1}{2}$, G= 

, $T=\begin{matrix} a & b \\ b & a. \end{matrix}$

Fig. 31.

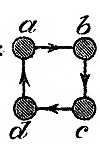
243. If $n=3$ we have $f=1$,

$(S)=\frac{1}{3}$, G= 

, $T=\begin{matrix} a & b & c \\ b & c & a \\ c & a & b. \end{matrix}$

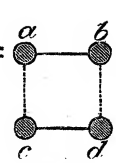
Fig. 32.

244. If $n=4$ we have $f=2$,

$(S)_1=\frac{1}{4}\cdot\frac{2}{2}$, $G_1=$ 

, $T_1=\begin{matrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c. \end{matrix}$

Fig. 33.

$(S)_2=\left(\frac{2}{2}\right)^3$, $G_2=$ 

, $T_2=\begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a. \end{matrix}$

Fig. 34.

245. If $n=5$ we have $f=1$,

$$(S)=\left(\frac{1}{5}\right)^2,\ G=$$

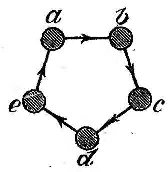


Fig. 35.

$$, T=\begin{array}{ccccc} a & b & c & d & e \\ b & c & d & e & a \\ c & d & e & a & b \\ d & e & a & b & c \\ e & a & b & c & d. \end{array}$$

246. If $n=6$ we have $f=2$,

$$(S)_1=\frac{1}{6}\cdot\frac{2}{3}\cdot\frac{3}{2},\ G_1=$$

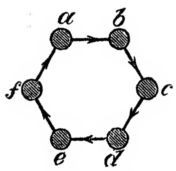


Fig. 36.

or

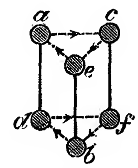


Fig. 37.

$$, T_1=\begin{array}{cccccc} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e. \end{array}$$

$$(S)_2=\frac{2}{3}\cdot\left(\frac{3}{2}\right)^3,\ G_2=$$

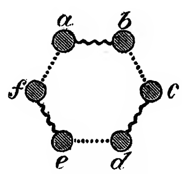


Fig. 38.

or

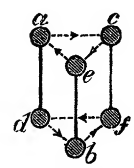


Fig. 39.

$$, T_2=\begin{array}{cccccc} a & b & c & d & e & f \\ b & a & f & e & d & c \\ c & d & e & f & a & b \\ d & c & b & a & f & e \\ e & f & a & b & c & d \\ f & e & d & c & b & a. \end{array}$$

247. If $n=7$ we have $f=1$,

$$(S)=\left(\frac{1}{7}\right)^3,\ G=$$

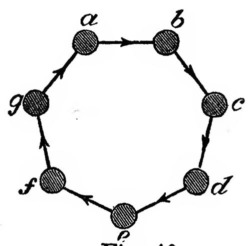


Fig. 40.

$$, T=\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \\ c & d & e & f & g & a & b \\ d & e & f & g & a & b & c \\ e & f & g & a & b & c & d \\ f & g & a & b & c & d & e \\ g & a & b & c & d & e & f. \end{array}$$

248. If $n=8$ we have $f=5$,

$$(S)_1=\left(\frac{4}{2}\right)^7,\ G_1=$$

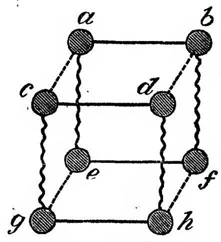


Fig. 41.

$$, T_1=\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a. \end{array}$$

$(\mathrm{S})_2=\left(\frac{4}{2}\right)^5\cdot\frac{2}{4}$, $\mathrm{G}_2=$

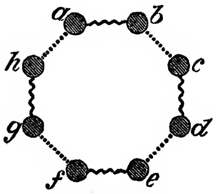


Fig. 42.

or

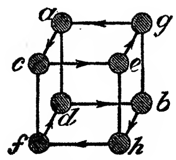


Fig. 43.

$$\mathrm{T}_2=\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & a & h & g & f & e & d & c \\ c & d & e & f & g & h & a & b \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & d & c & b & a & h & g \\ g & h & d & b & c & d & e & f \\ h & g & f & e & d & c & b & a. \end{array}$$

$(\mathrm{S})_3=\left(\frac{1}{8}\right)^2\frac{2}{4}\cdot\frac{4}{2}$, $\mathrm{G}_3=$

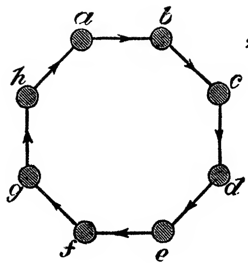


Fig. 44.

$$\mathrm{T}_3=\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & c & d & e & f & g & h & a \\ c & d & e & f & g & h & a & b \\ d & e & f & g & h & a & b & c \\ e & f & g & h & a & b & c & d \\ f & g & h & a & b & c & d & e \\ g & h & a & b & c & d & e & f \\ h & a & b & c & d & e & f & g. \end{array}$$

$(\mathrm{S})_4=\left(\frac{2}{4}\right)^2\cdot\left(\frac{4}{2}\right)^3$, $\mathrm{G}_4=$

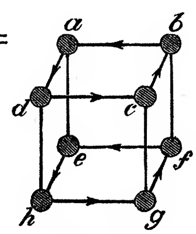


Fig. 45.

$$\mathrm{T}_4=\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & c & d & a & f & g & h & e \\ c & d & a & b & g & h & e & f \\ d & a & b & c & h & e & f & g \\ e & f & g & h & a & b & c & d \\ f & g & h & e & b & c & d & a \\ g & h & e & f & c & d & a & b \\ h & e & f & g & d & a & b & c. \end{array}$$

$(\mathrm{S})_5=\left(\frac{2}{4}\right)^3\cdot\frac{4}{2}$, $\mathrm{G}_5=$

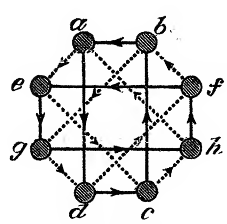


Fig. 46.

$$\mathrm{T}_5=\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & c & d & a & h & e & f & g \\ c & d & a & b & g & h & e & f \\ d & a & b & c & f & g & h & e \\ e & f & g & h & a & b & c & d \\ h & e & f & g & b & c & d & a \\ g & h & e & f & c & d & a & b \\ f & g & h & e & d & a & b & c. \end{array}$$

249. If $n=9$ we have $f=2$,

$$(\mathrm{S})_1=\left(\frac{1}{9}\right)^3\cdot\frac{3}{3},\quad \mathrm{G}_1=$$

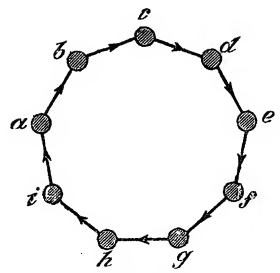


Fig. 47.

$$,\ \mathrm{T}_1=\begin{array}{ccccccccc}
a & b & c & d & e & f & g & h & i \\
b & c & d & e & f & g & h & i & a \\
c & d & e & f & g & h & i & a & b \\
d & e & f & g & h & i & a & b & c \\
e & f & g & h & i & a & b & c & d \\
f & g & h & i & a & b & c & d & e \\
g & h & i & a & b & c & d & e & f \\
h & i & a & b & c & d & e & f & g \\
i & a & b & c & d & e & f & g & h.
\end{array}$$

$$(\mathrm{S})_2=\left(\frac{3}{3}\right)^4,\quad \mathrm{G}_2=$$

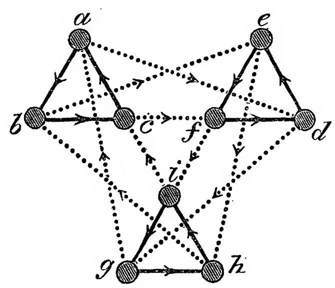


Fig. 48.

$$,\ \mathrm{T}_2=\begin{array}{ccccccccc}
a & b & c & d & e & f & g & h & i \\
b & c & a & e & f & d & h & i & g \\
c & a & b & f & d & e & i & g & h \\
d & e & f & g & h & i & a & b & c \\
e & f & d & h & i & g & b & c & a \\
f & d & e & i & g & h & c & a & b \\
g & h & i & a & b & c & d & e & f \\
h & i & g & b & c & a & e & f & d \\
i & g & h & c & a & b & f & d & e.
\end{array}$$

250. If $n=10$ we have $f=2$,

$$(\mathrm{S})_1=\left(\frac{1}{10}\right)^2\cdot\left(\frac{2}{5}\right)^2\cdot\frac{5}{2},\quad \mathrm{G}_1=$$

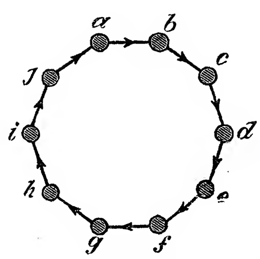


Fig. 49.

or

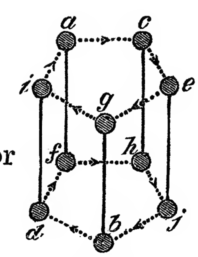


Fig. 50.

$$,\ \mathrm{T}_1=\begin{array}{cccccccccc}
a & b & c & d & e & f & g & h & i & j \\
b & c & d & e & f & g & h & i & j & a \\
c & d & e & f & g & h & i & j & a & b \\
d & e & f & g & h & i & j & a & b & c \\
e & f & g & h & i & j & a & b & c & d \\
f & g & h & i & j & a & b & c & d & e \\
g & h & i & j & a & b & c & d & e & f \\
h & i & j & a & b & c & d & e & f & g \\
i & j & a & b & c & d & e & f & g & h \\
j & a & b & c & d & e & f & g & h & i.
\end{array}$$

$(\mathrm{S})_2=\left(\frac{2}{5}\right)^2\cdot\left(\frac{5}{2}\right)^5$, $\mathrm{G}_2=$

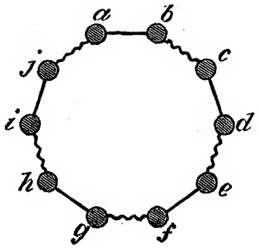


Fig. 51.

or

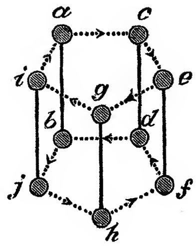


Fig. 52.

$$
,\ \mathrm{T}_2=\begin{array}{cccccccccc}
a & b & c & d & e & f & g & h & i & j \\
b & a & j & i & h & g & f & e & d & c \\
c & d & e & f & g & h & i & j & a & b \\
d & c & b & a & j & i & h & g & f & e \\
e & f & g & h & i & j & a & b & c & d \\
f & e & d & c & b & a & j & i & h & g \\
g & h & i & j & a & b & c & d & e & f \\
h & g & f & e & d & c & b & a & j & i \\
i & j & a & b & c & d & e & f & g & h \\
j & i & h & g & f & e & d & c & b & a.
\end{array}
$$

251. If $n=11$ we have $f=1$,

$(\mathrm{S})=\left(\frac{1}{11}\right)^5$, $\mathrm{G}=$

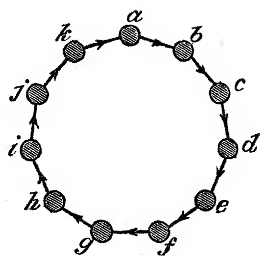


Fig. 53.

$$
,\ \mathrm{T}=\begin{array}{ccccccccccc}
a & b & c & d & e & f & g & h & i & j & k \\
b & c & d & e & f & g & h & i & j & k & a \\
c & d & e & f & g & h & i & j & k & a & b \\
d & e & f & g & h & i & j & k & a & b & c \\
e & f & g & h & i & j & k & a & b & c & d \\
f & g & h & i & j & k & a & b & c & d & e \\
g & h & i & j & k & a & b & c & d & e & f \\
h & i & j & k & a & b & c & d & e & f & g \\
i & j & k & a & b & c & d & e & f & g & h \\
j & k & a & b & c & d & e & f & g & h & i \\
k & a & b & c & d & e & f & g & h & i & j.
\end{array}
$$

252. If $n=12$ we have $f=5$,

$(\mathrm{S})_1=\left(\frac{1}{12}\right)^2\frac{2}{6}\cdot\frac{3}{4}\cdot\frac{4}{3}\cdot\frac{6}{2}$, $\mathrm{G}_1=$

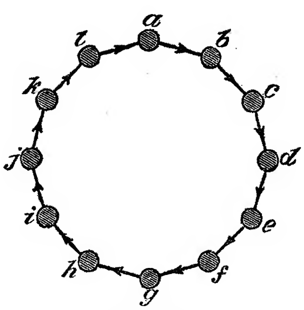


Fig. 54.

$$
,\ \mathrm{T}_1=\begin{array}{cccccccccccc}
a & b & c & d & e & f & g & h & i & j & k & l \\
b & c & d & e & f & g & h & i & j & k & l & a \\
c & d & e & f & g & h & i & j & k & l & a & b \\
d & e & f & g & h & i & j & k & l & a & b & c \\
e & f & g & h & i & j & k & l & a & b & c & d \\
f & g & h & i & j & k & l & a & b & c & d & e \\
g & h & i & j & k & l & a & b & c & d & e & f \\
h & i & j & k & l & a & b & c & d & e & f & g \\
i & j & k & l & a & b & c & d & e & f & g & h \\
j & k & l & a & b & c & d & e & f & g & h & i \\
k & l & a & b & c & d & e & f & g & h & i & j \\
l & a & b & c & d & e & f & g & h & i & j & k
\end{array}
$$

$(\mathrm{S})_2=\left(\frac{2}{6}\right)^3\cdot\frac{4}{3}\cdot\left(\frac{6}{2}\right)^3$, $\mathrm{G}_2=$

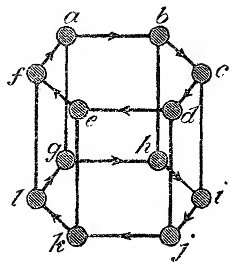
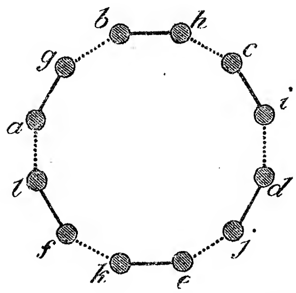


Fig. 55.

, $\mathrm{T}_2=$

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*a*	*h*	*i*	*j*	*k*	*l*	*g*
c	*d*	*e*	*f*	*a*	*b*	*i*	*j*	*k*	*l*	*g*	*h*
d	*e*	*f*	*a*	*b*	*c*	*j*	*k*	*l*	*g*	*h*	*i*
e	*f*	*a*	*b*	*c*	*d*	*k*	*l*	*g*	*h*	*i*	*j*
f	*a*	*b*	*c*	*d*	*e*	*l*	*g*	*h*	*i*	*j*	*k*
g	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*l*	*g*	*b*	*c*	*d*	*e*	*f*	*a*
i	*j*	*k*	*l*	*g*	*h*	*c*	*d*	*e*	*f*	*a*	*b*
j	*k*	*l*	*g*	*h*	*i*	*d*	*e*	*f*	*a*	*b*	*c*
k	*l*	*g*	*h*	*i*	*j*	*e*	*f*	*a*	*b*	*c*	*d*
l	*g*	*h*	*i*	*j*	*k*	*f*	*a*	*b*	*c*	*d*	*e*.

$(\mathrm{S})_3=\frac{2}{6}\cdot\frac{4}{3}\cdot\left(\frac{6}{2}\right)^7$, $\mathrm{G}_3=$



or

Fig. 56.

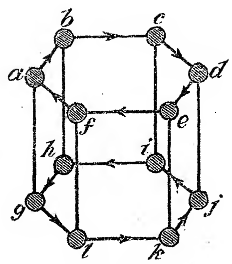


Fig. 57.

, $\mathrm{T}_3=$

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*a*	*h*	*i*	*j*	*k*	*l*	*g*
c	*d*	*e*	*f*	*a*	*b*	*i*	*j*	*k*	*l*	*g*	*h*
d	*e*	*f*	*a*	*b*	*c*	*j*	*k*	*l*	*g*	*h*	*i*
e	*f*	*a*	*b*	*c*	*d*	*k*	*l*	*g*	*h*	*i*	*j*
f	*a*	*b*	*c*	*d*	*e*	*l*	*g*	*h*	*i*	*j*	*k*
g	*l*	*k*	*j*	*i*	*h*	*a*	*f*	*e*	*d*	*c*	*b*
h	*g*	*l*	*k*	*j*	*i*	*b*	*a*	*f*	*e*	*d*	*c*
i	*h*	*g*	*l*	*k*	*j*	*c*	*b*	*a*	*f*	*e*	*d*
j	*i*	*h*	*g*	*l*	*k*	*d*	*c*	*b*	*a*	*f*	*e*
k	*j*	*i*	*h*	*g*	*l*	*e*	*d*	*c*	*b*	*a*	*f*
l	*k*	*j*	*i*	*h*	*g*	*f*	*e*	*d*	*c*	*b*	*a*.

$(\mathrm{S})_4=\left(\frac{3}{4}\right)^3\cdot\frac{4}{3}\left(\frac{6}{2}\right)^3$, $\mathrm{G}_4=$

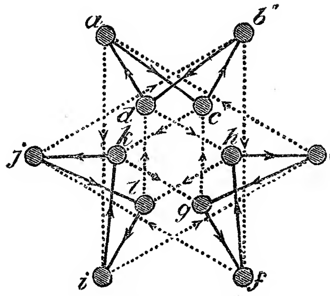


Fig. 58.

, $\mathrm{T}_4=$

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*a*	*d*	*c*	*j*	*i*	*l*	*k*	*f*	*e*	*h*	*g*
c	*d*	*b*	*a*	*g*	*h*	*f*	*e*	*k*	*l*	*j*	*i*
d	*c*	*a*	*b*	*l*	*k*	*i*	*j*	*h*	*g*	*e*	*f*
e	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*
f	*e*	*h*	*g*	*b*	*a*	*d*	*c*	*j*	*i*	*l*	*k*
g	*h*	*f*	*e*	*k*	*l*	*j*	*i*	*c*	*d*	*b*	*a*
h	*g*	*e*	*f*	*d*	*c*	*a*	*b*	*l*	*k*	*i*	*j*
i	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
j	*i*	*l*	*k*	*f*	*e*	*h*	*g*	*b*	*a*	*d*	*c*
k	*l*	*j*	*i*	*c*	*d*	*b*	*a*	*g*	*h*	*f*	*e*
l	*k*	*i*	*j*	*h*	*g*	*e*	*f*	*d*	*c*	*a*	*b*.

$(\mathrm{S})_5=\binom{4}{3}^4\cdot\binom{6}{2}^3$, $\mathrm{G}_5=$

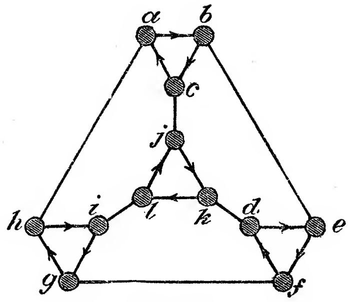


Fig. 59.

$$, \mathrm{T}_5=\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l \\ b & c & a & l & j & k & d & e & f & h & i & g \\ c & a & b & g & h & i & l & j & k & e & f & d \\ d & e & f & a & b & c & j & k & l & g & h & i \\ e & f & d & i & g & h & a & b & c & k & l & j \\ f & d & e & j & k & l & i & g & h & b & c & a \\ g & h & i & c & a & b & e & f & d & l & j & k \\ h & i & g & k & l & j & c & a & b & f & d & e \\ i & g & h & e & f & d & k & l & j & a & b & c \\ j & k & l & f & d & e & b & c & a & i & g & h \\ k & l & j & h & i & g & f & d & e & c & a & b \\ l & j & k & b & c & a & h & i & g & d & e & f. \end{array}$$

Some General Forms of Groups.

253. The graphical representations of the preceding forms suggest others which exist in the case of larger numbers of units. Thus the form in which a simple circuit passes through all the units of the group appears for all the given values of n; it obviously also exists as one form of group for every other value of n.

254. So if n be even, a form such as that given by the first figure of G_2 when $n=4$, 6, 8, and 10, and by the first figure of G_3 when $n=12$, in which all the units are connected by a continuous chain of non-polar lines of two kinds, clearly exists whatever value n has.

255. If $n=2^m$ we have the form such as G when $n=2$

G_2	,,	4
G_1	,,	8

in which all pairs are symmetrical. This form is closely related to the important one considered in Logic (sec. 381), which may be derived from it by ignoring the differences between pairs constituting a pure complete network.

*A Family of Groups.**

256. Let us consider groups in which every circuit is of period 2 or 4. Some of the symmetrical pairs of these groups are component pairs of sets of four units composing simple circuits of period 4, say are *diagonal pairs* ($a\,e$ in fig. 43 is a diagonal pair); while some (*e.g.*, cf in fig. 43) are not diagonal pairs of any simple circuit. We might study these groups generally, but I propose here to restrict myself to groups in which all diagonal pairs are undistinguished from each other, so that they all belong to the same one-way simple network, say the *diagonal network.*

* The investigation of sections 256–269 was suggested by the late Professor CLIFFORD'S paper on "GRASSMAN'S Extensive Algebra," in the 'American Journal of Mathematics,' vol. i., pp. 350–358.

257. Every unit a is accompanied by one unit a', which makes with it a diagonal pair. We may term a' the *companion* of a. We have $(a')'=a$.

258. Let each pair of the diagonal network be represented by a π, then if α be an unsymmetrical pair of any simple network of unsymmetrical pairs, we have

$$\alpha^2=\pi, \quad \alpha^3=\alpha', \quad \alpha^4=1.$$

Also if β be a symmetrical pair we have $\beta^2=1$.

259. The network (π) is commutative with all the others of the group. For if α be any unsymmetrical pair we have $\alpha^2=\pi$, and therefore $\alpha\pi=\alpha\alpha\alpha=\pi\alpha$; and if β be any symmetrical pair, either there is no unsymmetrical pair in the group, every circuit being of period 2, in which case every pair is commutative with every other, or else there are unsymmetrical pairs $\lambda\lambda \ldots \mu\mu \ldots$, &c., in which case let

$$\beta\gamma=\lambda, \quad \gamma\beta=\mu,$$

then

$$\pi\beta=\lambda^2\beta=\beta\gamma.\beta\gamma.\beta=\beta.\gamma\beta.\gamma\beta=\beta\mu^2=\beta\pi.$$

260. If α, β be any pairs, $\alpha\beta=\beta\alpha$ or $\beta\alpha\pi$. For let $\alpha\beta=\lambda$, and $\alpha\beta=\beta\alpha\sigma$, then

$$\begin{aligned}\lambda^2=\alpha\beta.\alpha\beta=\alpha\beta\beta\alpha.\sigma=\alpha 1\alpha.\sigma \text{ or } \alpha\pi\alpha.\sigma=\alpha^2\sigma \text{ or } \alpha^2\pi\sigma \\ =\sigma \text{ or } \pi\sigma.\end{aligned}$$

But $\lambda^2=\pi$ or 1, therefore $\sigma=\pi$ or 1.

261. We have

$$\begin{aligned}\alpha\beta\gamma\delta&=\alpha\gamma\beta\delta \text{ or } \alpha\gamma\beta\pi\delta \\ &=\alpha\gamma\beta\delta \text{ or } \alpha\gamma\beta\delta\pi\end{aligned}$$

according as

$$\beta\gamma=\gamma\beta \text{ or } \gamma\beta\pi$$

i.e., if we start from any given unit and proceed along a chain composed of pairs belonging to a given collection of networks, and then starting from the same unit proceed along a chain composed of pairs of the same kind but in a different order, we shall either arrive at the same unit as in the previous case or at its companion.

262. If we substitute for π the symbol -1, so that

$$a'=\pi a=-a$$

we have, if β be a symmetrical pair, $\beta^2=1$; if unsymmetrical, $\beta^2=-1$. Also, if α and β be any two pairs, we have

$$\alpha\beta=\beta\alpha \text{ or } -\beta\alpha.$$

263. The group (No. 5 of sec. 248 ; see fig. 46*b*) of 8 units given by the following table :—

u	x'	y'	z'	u'	x	y	z
x'	u'	z	y'	x	u	z'	y
y'	z'	u'	x	y	z	u	x'
z'	y	x'	u'	z	y'	x	u
u'	x	y	z	u	x'	y'	z'
x	u	z'	y	x'	u'	z	y'
y	z	u	x'	y'	z'	u'	x
z	y'	x	u	z'	y	x'	u'

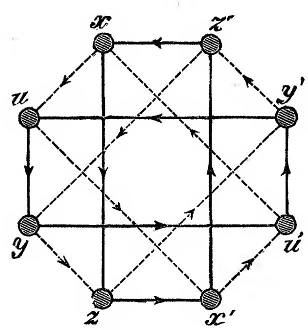


Fig. 46*b*.

is one of the species we are considering. The laws of multiplication of the component simple networks (sec. 234), viz. of

$$(ux),\ (ux'),\ (uy),\ (uy'),\ (uz),\ (uz'),\ (uu),\ (uu'),$$

(sec. 201) are the same as those of the quaternion expressions

$$i,\ -i,\ j,\ -j,\ k,\ -k,\ 1,\ -1.$$

For example, we have uy>——<xz and uz>——<zu', and thus, just as we have $ijk=-1$, so we have $(ux)(uy)(uz)=(ux)(xz)(zu')=(uu')$. I shall accordingly term the group a *quaternion* group.

264. A double network of which the detached portions (sec. 206) are each such as that shown in fig. 46*b* may be termed a *pure quaternion network*. The two component networks are non-commutative; each may be termed the *conjugate* of the other, and we may represent the conjugate of (α) by $(\bar{\alpha})$.

265. I proceed to show that every group of the family now under consideration may be defined by a pure complete network (sec. 235) consisting of m pairs of conjugate networks (*i.e.* m distinct quaternion networks), and r other simple networks having no conjugates in the network, ($2m+r$ simple networks in all) ; the $2m+r$ simple networks being such that each two are commutative unless they be a conjugate pair.

266. Consider a pure compound network consisting of m distinct pure quaternion networks. If P be any chain of the pairs composing the $2m$ simple networks, each sort of pair entering only once, if at all, into the chain, we have $\mathrm{P}\alpha=\alpha\mathrm{P}\pi$ or $\alpha\mathrm{P}$, *i.e.*, P non-commutative or commutative with α, according as the chain contains or does not contain a pair $\bar{\alpha}$. Now if (ν) be any simple network which is not one of the $2m$. (ν) will be commutative with some of the $2m$, and non-commutative with others. Let N be a chain containing one pair from each of the conjugates of those of the $2m$ networks with which (ν) is non-commutative, then the simple network $(\mathrm{N}\nu)$ (sec. 233)

is commutative with each of the $2m$. For if (α) one of the $2m$ is non-commutative with (ν), the chain N contains the pair $\bar{\alpha}$, and thus (α) is non-commutative with (N) also, and therefore commutative with $(\mathrm{N}\nu)$; and if (α) is commutative with (ν), N does not contain $\bar{\alpha}$, and thus (α) is commutative with (N) also, and therefore also with $(\mathrm{N}\nu)$.

267. Suppose now we have any pure complete network consisting of $2m$ networks such as the $2m$ of the preceding sections, and also of other simple networks (λ), (μ), (ν) . . . ; we can substitute for the latter the simple networks $(\mathrm{L}\lambda)$, $(\mathrm{M}\mu)$, $(\mathrm{N}\nu)$ . . . where the (L), (M), (N) . . . are networks such as the (N) of the last section, and we shall still have a pure complete network, and it will be such that the networks other than the $2m$ are commutative with the $2m$. Now these networks other than the $2m$ are either all commutative with each other, or else there are two at least which are not; in the latter case we may add such two to the $2m$ and get $2m+2$ such as the $2m$; we may then as before substitute for the remainder simple networks which are all commutative with the $2m+2$, and may repeat the process continually until the remaining networks other than the conjugate pairs are all commutative with each other.

268. Any group of the family we are considering is such that all the simple networks are commutative, or that two at least are not; in the latter case, if we take the two as conjugates, and take in others so as to constitute a pure complete network, we can proceed to deal with this in the mode we have just indicated. Thus in every case we can obtain a pure complete network containing n lots of two conjugate networks, and one lot of r commutative networks, the networks of each of the $(n+r)$ lots being commutative with those of the others. Either n or r may vanish.

269. If we make $n=4$ and $r=1$ we have a group of 16 units given by the following table :—

u	x'	y'	z'	u'	x	y	z	U	X′	Y′	Z′	U′	X	Y	Z
x'	u'	z	y'	x	u	z'	y	X′	U′	Z	Y′	X	U	Z′	Y
y'	z'	u'	x	y	z	u	x'	Y′	Z′	U′	X	Y	Z	U	X′
z'	y	x'	u'	z	y'	x	u	Z′	Y	X′	U′	Z	Y′	X	U
u'	x	y	z	u	x'	y'	z'	U′	X	Y	Z	U	X′	Y′	Z′
x	u	z'	y	x'	u'	z	y'	X	U	Z′	Y	X′	U′	Z	Y′
y	z	u	x'	y'	z'	u'	x	Y	Z	U	X′	Y′	Z′	U′	X
z	y'	x	u	z'	y	x'	u'	Z	Y′	X	U	Z′	Y	X′	U′
U	X′	Y′	Z′	U′	X	Y	Z	u	x'	y'	z'	u'	x	y	z
X′	U′	Z	Y′	X	U	Z′	Y	x'	u'	z	y'	x	u	z'	y
Y′	Z′	U′	X	Y	Z	U	X′	y'	z'	u'	x	y	z	u	x'
Z′	Y	X′	U′	Z	Y′	X	U	z'	y	x'	u'	z	y'	x	u
U′	X	Y	Z	U	X′	Y′	Z′	u'	x	y	z	u	x'	y'	z'
X	U	Z′	Y	X′	U′	Z	Y′	x	u	z'	y	x'	u'	z	y'
Y	Z	U	X′	Y′	Z′	U′	X	y	z	u	x'	y'	z'	u'	x
Z	Y′	X	U	Z′	Y	X′	U′	z	y'	x	u	z'	y	x'	u'

The component simple networks of this group, viz. :—

$$(ux),\quad (ux'),\quad (uy),\quad (uy'),\quad (uz),\quad (uz'),\quad (uu),\quad (uu'),$$
$$(u\mathrm{X}),\quad (u\mathrm{X}'),\quad (u\mathrm{Y}),\quad (u\mathrm{Y}'),\quad (u\mathrm{Z}),\quad (u\mathrm{Z}'),\quad (u\mathrm{U}),\quad (u\mathrm{U}'),$$

are subject to the same laws of multiplication as the bi-quaternion expressions

$$i,\quad -i,\quad j,\quad -j,\quad k,\quad -k,\quad 1,\quad -1,$$
$$\omega i,\quad -\omega i,\quad \omega j,\quad -\omega j,\quad \omega k,\quad -\omega k,\quad \omega,\quad -\omega.$$

R-adic Groups.

270. We may have a set such that the unified component r-ads constitute a group. We may call the set an *r-adic group.* If $r=1$ the set is an ordinary group.

271. In an r-adic group any r-ad may be made to correspond to any other r-ad, but the correspondence of two r-ads completely determines a self-correspondence of the set. Thus if the form of the set be known, any self-correspondence is fully represented by a two-row r-column constituent of the table.

272. In an r-adic group of n units the tabular representation has $\frac{|\underline{n}}{|\underline{n-r}}$ rows. For all $(r-1)$-ads are undistinguished, and there are $\frac{|\underline{n}}{|\underline{n-r+1}}$ of these, and while any one remains identically-correspondent the remaining $(n-r+1)$-ad has self-correspondences characteristic of a group, *i.e.*, has $(n-r-1)$ self-correspondences. Thus there are in all $(n-r-1)\frac{|\underline{n}}{|\underline{n-r-1}}=\frac{|\underline{n}}{|\underline{n-r}}$ self-correspondences of the group including the identical correspondence.

273. A correspondence of two undistinguished aspects of an r-adic group has not more than $r-1$ foci (sec. 122).

274. Let S be a system of which the units are the whole collection of points lying on a straight line, viz., $a, b, c, d, \ldots$. Any aspect $abcd\ldots$ of S is what is usually termed the "range $abcd\ldots$." For the range $abcd\ldots$ we can by a homographic transformation substitute another range of the same points $a, b, c, d, \ldots$, *i.e.*, another aspect of S. Employing all the various homographic transformations we get a set of aspects of S. Now, if we assume that this set is a single system, *i.e.*, that all aspects of S derived from $abcd\ldots$ by homographic transformations are undistinguished from $abcd\ldots$ and each other, but are distinguished from all aspects which cannot be derived from $abcd\ldots$ by such transformations; then S will be a triadic group. For we can by a proper homographic transformation substitute for any three units of S any other three units of S, but when this substitution is made, every other unit of S has a definite unit of S substituted for it.

Substitutions.

275. Each row of the tabular representation of a system is derived from each of the others by definite substitutions. Instead of writing down the various rows, we may give one only, and state the law or laws according to which the other rows are derived. Thus the form of a system $a, b, c, d, \ldots$ is given by merely writing down the letters representing the units of the system (since each letter appears only once, we may represent the units by the letters in place of their sorts), and stating the laws of substitution, in other words, the substitutions proper to the form (sec. 105).

276. Conversely, in considering a system of n letters, or other things, admitting of certain substitutions, we are considering a system of n units of a definite form.

277. The various arrays of letters considered when dealing with substitutions are thus aspects of a system.

278. When the arrays are regarded as units the substitutions will be represented by pairs; substitutions of the same sort by undistinguished pairs; similar substitutions by similar pairs (sec. 230).

279. The substitution of one aspect of a system for another, substitutes for a component collection of any form another component collection of the same form.

Algebras.

280. Consider the system V of $3n$ units arrived at by regarding as units the pairs connecting the units $a, b, c, d, \ldots$ of a system S of n units with the three distinguished units λ, μ, ν. If we adopt the method of representation of secs. 139 and 152 the units of V will be represented by—

$$\begin{array}{cccccc} (a\lambda) & (b\lambda) & (c\lambda) & . & . & \\ (a\mu) & (b\mu) & (c\mu) & . & . & \&c. \\ (a\nu) & (b\nu) & (c\nu) & . & . & \end{array}$$

It will, however, conduce to clearness in the present instance to write them thus—

$$\begin{array}{cccccc} a_\lambda & b_\lambda & c_\lambda & . & . & \\ a_\mu & b_\mu & c_\mu & . & . & \&c. \\ a_\nu & b_\nu & c_\nu & . & . & \end{array}$$

The symbols of the three rows represent the units of three different systems, each of which systems is a replica of the others and of S. We may denote the three systems when regarded as units by S_λ, S_μ, S_ν, respectively.

281. Now consider the system E arrived at by regarding the triads connecting the three systems as units. This will contain n^3 units, which may be written thus—$(a_\lambda a_\mu a_\nu)$, $(a_\lambda b_\mu c_\nu)$, &c.

282. We have collections of triads connecting $S_\lambda S_\mu S_\nu$* in which each pair connecting S_λ and S_μ appears once in a triad, and once only, *e.g.*, if S contains two units a, b, only, we have such collections as

$$\begin{matrix} a_\lambda\, a_\mu\, a_\nu \\ a_\lambda\, b_\mu\, a_\nu \\ b_\lambda\, a_\mu\, a_\nu \\ b_\lambda\, b_\mu\, a_\nu \end{matrix} \quad \text{or} \quad \begin{matrix} a_\lambda\, a_\mu\, b_\nu \\ a_\lambda\, b_\mu\, a_\nu \\ b_\lambda\, a_\mu\, b_\nu \\ b_\lambda\, b_\mu\, a_\nu \end{matrix} \quad \text{or} \quad \begin{matrix} a_\lambda\, a_\mu\, b_\nu \\ a_\lambda\, b_\mu\, b_\nu \\ b_\lambda\, a_\mu\, b_\nu \\ b_\lambda\, b_\mu\, a_\nu \end{matrix}$$

The number of such collections in the general case is n^{n^2}, each containing n^2 triads. When the triads are regarded as units we get n^{n^2} collections each containing n^2 units of E.

283. When these collections are unified we get a system A of n^{n^2} units, which will be a multiple system.

284. The form of A is in part independent of that of S and in part dependent, certain components being distinguished whatever be the form of S, while it depends on what the form of S is whether others are distinguished or not.

285. When considering the multiplicity of the system A, or those peculiarities of form it possesses which arise from the mode of its construction apart from any special peculiarities in the form of S, we regard S as a single heap system.

286. If S be a single heap system, certain units of A are undistinguished from each other, while others are distinguished. If S be not a single heap system, so that its self-correspondences are more restricted than they would be in the former case, some units of A originally undistinguished become distinguished, so that a unit which was one of several undistinguished units, may become unique. Thus certain units of A have definite relations to S, and are of use as auxiliaries to the latter.

287. A convenient mode of conceiving of the system E is to regard its units as small cubes composing a big one, the axes of the cube being represented by λ, μ, and ν. Any layer of the cube normal to λ, say a λ layer, contains small cubes representing triads (unified) all containing the same unit of S_λ, and similarly in the cases of μ and ν. If a_λ be the unit of S_λ in the triads which are represented when unified by a λ layer, the layer may be termed the a_λ layer, and when unified represents a_λ. We may also conveniently term a row of cubes parallel to the axis λ a $\mu\nu$ row; it contains cubes representing triads all of which contain the same connecting pair of S_μ and S_ν, say the pair $a_\mu b_\nu$, and when unified is represented by $(a_\mu b_\nu)$. The λ layers thus represent the system S_λ, and must be supposed to admit of substitutions among themselves characteristic of the form of S_λ (sec. 276), and similarly in the case of μ and ν. Here the collection of units, which when unified gives a unit of A, is represented by cubes

* In strictness this should be "triads connecting the systems which when unified are S_λ, S_μ, S_ν respectively;" but the abbreviation is convenient and will not lead to misconception (see sec. 20).

such that each $\lambda\mu$ row contains one cube and one only of the collection. If S be a single heap system, the multiplicity of A will be the number of the systems into which the collections of cubes thus constituted break up.

288. We may represent any one of the collections which when unified is a unit of A by a square diagram. Take, for example, the case $n = 4$; we have such a diagram as the following:—

$$\begin{array}{c|c|c|c|c|}
 & a & b & c & d \\ \hline
a & a & d & b & d \\ \hline
b & c & c & a & b \\ \hline
c & b & a & c & d \\ \hline
d & d & b & a & a \\ \hline
\end{array}$$

Here the square may be supposed to be a face of the cube normal to ν, λ and μ being supposed to be vertical and horizontal respectively. The left hand letters indicate whether the adjacent λ layers are a_λ or b_λ or c_λ, &c., respectively, the top letters whether the adjacent μ layers are a_μ or b_μ or c_ν &c., respectively; and the letters contained in the squares, each of which squares is supposed to represent a cube, *i.e.*, a unit of E, indicate which ν layers the various cubes lie in.

289. By regarding the pairs connecting the system E and a system of units π, ω, ξ, &c., which are all distinguished from each other, as unified, we may get a number of systems E_π, E_ω, E_ξ, &c., replicas of each other and of E, and we may consider systems arrived at by taking one or more component systems of each of the systems E_π, E_ω, E_ξ, &c. Similarly in the case of A.

290. Now let α be one of the units of A arrived at by regarding as unified a collection of units of E of which $(a_\lambda b_\mu c_\nu)$ is one. This collection of units of E which when unified is α, contains no other unified triad of the sort $(a_\lambda b_\mu \cdot)$, *e.g.*, $(a_\lambda b_\mu d_\nu)$, so that c_ν is unique with respect to α, a_λ, b_μ; for when c_ν is changed to d_ν, a_λ and b_μ remaining unchanged, α is changed to a unit arrived at by regarding a collection of units of E containing $(a_\lambda b_\mu d_\nu)$ as unified, which cannot be α. We may, therefore, write

$$c_\nu = (\alpha a_\lambda b_\mu).$$

We may deal with the other triads concerned in arriving at α in the same way, and get a collection of n^2 equations, viz.—

$$\begin{array}{lll}
(\alpha a_\lambda a_\mu) = \cdot & (\alpha b_\lambda a_\mu) = \cdot & \\
(\alpha a_\lambda b_\mu) = c_\nu & (\alpha b_\lambda b_\mu) = \cdot & \text{\&c.} \\
(\alpha a_\lambda c_\mu) = \cdot & (\alpha b_\lambda c_\mu) = \cdot & \\
\text{\&c.} & \text{\&c.} & \text{\&c.}
\end{array}$$

where the letters on the right-hand side of the equations depend on the particular unit α selected.

291. Now we may make $c_\nu \equiv c$, and similarly in the case of every other unit of S and S_ν; also we may represent the units λ and μ by two *positions*, and write $a_\lambda b_\mu$ thus $a_1 b_2$, where the order of a_1 and b_2 is immaterial, or merely ab where the order is material; the equations then become

$$
\begin{array}{lll}
(\alpha aa) = . & (\alpha ba) = . & \\
(\alpha ab) = c & (\alpha bb) = . & \&c. \\
(\alpha ac) = . & (\alpha bc) = . & \\
\&c. & \&c. & \&c.
\end{array}
$$

If then we have two equations such as

$$(\alpha ab) = c \text{ and } (\alpha cd) = e,$$

we may write

$$e = (\alpha(\alpha ab)d),$$

and similarly in other cases; so that we obtain complex expressions representing the form relations which the units of S hold to each other and to the units λ, μ, α; which expressions, however, admit of considerable simplification in certain cases.

292. An equation such as $(\alpha ab) = c$ I shall term a *primitive equation.* The whole collection of triads which are concerned in arriving at α will be termed a *primitive algebra*, α being termed a *unified primitive algebra.** An equation, such as $e = (\alpha(\alpha ab)d)$, where only one primitive algebra is involved, may be termed a *complex primitive equation.*

293. Those components of E which are represented when unified by the units of A furnish us, by the application of the preceding methods, with every possible primitive algebra, associative or non-associative, commutative or non-commutative, &c., and we can discuss the number of the forms, and the relations of the various algebras, by discussing E in the case in which S is a single heap system (sec. 285).

294. We may have any number of unified primitive algebras α, β, γ, δ, . . . which may be undistinguished from each other or not. If the unified primitive algebras are all selected from one system A_π (sec. 289), they may be said to be algebras of the *same operation* π, if from different systems A_π, A_ω, they may be said to be algebras of different operations, *e.g.*, multiplication, addition, &c.

295. We may deal with each of these algebras as with α and we may also deal with them in combination and obtain equations such as $e = (\alpha(\beta ab)d)$, giving complicated expressions for the units of S, which will represent the form relations they hold to each other and to the units λ, μ, α, β, &c.

* We might regard the primitive equations as units, and apply the term "unified primitive algebra," not to α, but to the unit arrived at by regarding as unified the collection of unified primitive equations derived from the triads concerned in arriving at α.

296. Where two or more primitive algebras are considered, we may speak of the unit arrived at by regarding them as unified as a *unified compound algebra.*

297. The discussion of form is very generally carried on by the help of these auxiliary algebras. In some cases the units a, b, c, d, . . . of the base system under consideration are regarded as unified pairs Xa', Xb', Xc', ... where X is an algebra primitive or compound, and a', b', c', . . . are units constituting a single heap system, which have the correspondences characteristic of the system a, b, c, . . . as long as X remains identically correspondent. But various devices exist which must be examined in each case.

298. Where only one unified primitive algebra is considered, we may regard the unit α as expressed by the brackets ( ), and write an equation $(\alpha ab)=c$ simply as $(ab)=c$, or we may omit the brackets and write the equation thus $ab=c$.

299. If in an algebra every equation $(ab)=c$ is accompanied by an equation $(ba)=c$, the algebra is said to be *commutative.*

300. If in an algebra any three equations such as

$$\begin{gathered}(ab)=x\\(bc)=y\\(xc)=z\end{gathered}$$

are accompanied by the equation

$$(ay)=z$$

the algebra is said to be *associative.* Since $((ab)\ c)=(a(bc))$ either expression may be written (abc), and no ambiguity arises.

301. If π and ρ two algebras are such that the equations

$$\begin{array}{ll}(\pi ab)=x & (\pi ba)=l\\(\pi ac)=y & (\pi ca)=m\\(\pi ad)=z & (\pi da)=n\end{array}$$
$$(\rho bc)=d$$

are accompanied by the equations

$$(\rho xy)=z \qquad (\rho lm)=n$$
$$\text{i.e., if } \rho\ (\pi ab\ .\ \pi ac)=\pi\ (a\ .\ \rho bc)$$
$$\text{and } \rho\ (\pi ba\ .\ \pi ca)=\pi\ (\rho bc\ .\ a)$$

then the algebra π is said to be *distributive* with respect to the algebra ρ.

We may take πab to be $a\times b$, and ρab to be $a+b$; we then have

$$\begin{gathered}a\times b+a\times c=a(b+c)\\b\times a+c\times a=(b+c)\times a\end{gathered}$$

302. An algebra may be *self-distributive*, *i.e.*, such that

$$(a(bc))=((ab)(ac)).$$

(See sec. 349.)

303. An important class of primitive algebras is that in which $(ab) \neq a$ or b, except in the case of one unit z, which is such that $(zz)=z$, and also such that whatever unit of S a may be $(za)=a$ and $(az)=a$. Here every unit a is accompanied by another a' such that $(aa')=z$. Ordinary addition and multiplication furnish two such algebras. In the former (ab) is $a+b$, a is 0, and a' is $-a$; in the latter (ab) is $a \times b$, z is 1, and a' is $\frac{1}{a}$.

304. The algebras which are such that in the case of each unit a we have $(aa)=a$, are also of considerable importance. We have such an algebra in the case of ordinary logic (sec. 360).

305. We may in ordinary parlance speak of three units a_λ, b_μ, c_ν, composing the triad giving rise to a primitive equation, as if they were units a, b, c, of S, without any ambiguity arising. The unit a when dealt with as a multiplier will be a_λ, and so on; but it will not be necessary to be continually pointing this out. We may thus speak of products (aa) when there is really only one a.

306. If in any system of units S every pair is accompanied by one or more units unique with respect to the pair, we may select one of these accompanying units in the case of each pair and term it the *product* of the pair. Where there is only one such accompanying unit in the case of each pair, this will be the product, but if there be several such, any one of these might be chosen as the product. The most natural mode of selection in the case of undistinguished pairs is to select products which will with the pairs make undistinguished triads. (Secs. 343, 349.)

307. In the case of distinguished pairs we cannot of course do this, but we may in some cases choose products so that the resulting triads will all be of the same form, this not being so if other products be chosen; or we may choose the products, so that by ignoring some difference the triads all become undistinguished from each other, the products still being unique with respect to the pairs.

308. In cases in which some pairs are not accompanied by a unit unique with respect to them, or even in other cases, we may add to S a single system of one unit Z, and call this the product of the pairs; Z being also considered as the product of each unit of S when multiplied by or into Z. (Sec. 319, and *cf.* sec. 343.)

309. In certain cases the products considered may not be unique with respect to the multiplier and multiplicand, but with respect to a collection containing them and certain other units, which remain the same in the case of every multiplier and multiplicand of the system, and may therefore be termed *constants.*

An instructive example of this is furnished by a system of collinear points. If i, z, u, a, b, be five points of such a system, there is one point, and one only, which is such that the six points

$$i,\ z,\ u,\ c,\ a,\ b,$$

lie respectively on the six lines

$$PQ,\ RS,\ PR,\ QS,\ PS,\ QR,$$

passing through four coplanar points P, Q, R, S : *i.e.*, the unit c is unique with respect to the collection i, z, u, a, b. Now let i, z, u, be three *constants* in the foregoing sense. Then we may call c the *product* of a and b. The primitive algebra which thus arises may readily be shown to be associative and commutative.

Again, there is one point d, and one only, which is such that the points

$$i,\ i,\ z,\ d,\ a,\ b,$$

lie respectively on the six lines

$$LM,\ NO,\ LN,\ MO,\ LO,\ MN,$$

passing through four coplanar points L, M, N, O : *i.e.*, the unit d is unique with respect to the collection i, z, a, b. Here, if i and z be constants, d may be regarded as a second species of product of a and b, and may be termed the *sum* of a and b. The primitive algebra thus arising is also associative and commutative.

The first of these two primitive algebras is distributive as regards the second. In fact the compound algebra composed of the two primitive algebras is of the same kind as the ordinary algebra of quantity, the units i, z, u, corresponding respectively to the ∞, 0, and 1 of such algebra.*

310. We may speak of the collection consisting of the products of the pairs connecting two component collections of a system S as the product of those collections. If the number of units in the multiplier collection be n, and the number of those in the multiplicand be m, the number in the product will not necessarily be mn, for the product x of a connecting pair ap may be the same unit as the product of another connecting pair bq. We may speak of the collection consisting of the products of all the pairs of a collection a, b, c, d .... together with the products (aa), (bb), (cc) ... as the square of the collection a, b, c, d .... In the same way we may have cubes, &c., of a collection.

311. If we substitute for S the system S_{H} of sec. 191, we may regard products (ap) and (bq) as two different units (sec. 192), and if this be done in the case of each product, we shall have mn units in the product.

312. We may regard the components of S or S_{H} as units, and then the products will be units, the products of pairs of units.

Quadrates.

313. Let S_1, S_2, be two single heap systems, replicas of each other. Let a_1, b_1, c_1 ... be units of S_1; a_2, b_2, c_2 .... be corresponding units of S_2. Considering the pairs connecting S_1 and S_2; these may be written

* See a paper by the writer "On an Extension of Ordinary Algebra" in the *Messenger of Mathematics*. Vol. XV. New Series, p. 188.

$$\begin{array}{lll} a_1\ a_2 & a_1\ b_2 & a_1\ c_2, \text{ \&c.} \\ b_1\ a_2 & b_1\ b_2 & b_1\ c_2, \text{ \&c.} \\ c_1\ a_2 & c_1\ b_2 & c_1\ c_2, \text{ \&c.} \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

If it be understood that we are throughout dealing with pairs connecting S_1S_2 we may write any aspect $p_1\ q_2$ thus pq, the position of a letter in the pair taking the place of the unit represented by the subscript number. We then have under consideration the aspects

$$\left.\begin{array}{lll} aa & ab & ac, \text{ \&c.} \\ ba & bb & bc, \text{ \&c.} \\ ca & cb & cc, \text{ \&c.} \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}\right\} \ldots\ldots (\text{X})$$

These will be n^2 in number, where n is the number of units in each of the systems S_1 S_2. The aspects aa, bb, cc, . . . are undistinguished from each other, but are distinguished from all the other aspects ab, ac, bc . . . . which again are all undistinguished from each other.*

314. We may regard the aspects as unified, and shall then get a double system P composed of the system P_1 of n unified aspects (aa), (bb), (cc) . . . . . , and the system P_2 of n^2-n unified aspects (ab), (ac), (bc) . . . . †

315. The mode of arranging the aspects adopted at X in sec. 313 suggests a mode of clothing the units of P which is very convenient for purposes of description, and for emphasizing the peculiarities of P; viz., that of regarding the units as arranged in rows and columns, the order of which is to be disregarded, so that the rows as units form a single heap system, as also the columns.

316. In fig. 60 we have a system P of the species considered, in which $n=4$; the asterisks represent units of P_1, the dots those of P_2.

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \\ \cdot & \cdot & * & \cdot \\ \cdot & \cdot & \cdot & * \end{array}$$

Fig. 60.

317. The pairs of P form a system of multiplicity 13; viz., we have the 13 sorts of pairs given by the following diagrams, where in the last seven cases the top unit in the first column, and the bottom in the second are those composing the pair considered.

(1) $* \ \cdot$, (2) $\cdot \ *$, (3) $\cdot \ \cdot$, (4) $\begin{smallmatrix} * \\ \cdot \end{smallmatrix}$, (5) $\begin{smallmatrix} \cdot \\ * \end{smallmatrix}$, (6) $\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}$,

(7) $\begin{smallmatrix} * & \cdot \\ \cdot & * \end{smallmatrix}$, (8) $\begin{smallmatrix} * & \cdot \\ \cdot & \cdot \end{smallmatrix}$, (9) $\begin{smallmatrix} \cdot & \cdot \\ \cdot & * \end{smallmatrix}$, (10) $\begin{smallmatrix} \cdot & * \\ * & \cdot \end{smallmatrix}$, (11) $\begin{smallmatrix} \cdot & * \\ \cdot & \cdot \end{smallmatrix}$, (12) $\begin{smallmatrix} \cdot & \cdot \\ * & \cdot \end{smallmatrix}$, (13) $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}$.

* If we confine ourselves to the latter system of aspects, we might regard them as aspects of either of the systems S_1 and S_2; but the aspects aa, bb, cc . . . . must be regarded as connecting pairs of the two systems S_1 and S_2.

† We should obtain a system of the same form as P if we took S_1, and the unified pairs of S_1. Here S_1 takes the place of P_1, and the unified pairs of S take the place of P_2.

The pairs (1), (5), (10), and (12) will be termed *joined* pairs; each is such that the column in which the first unit lies and the row in which the second unit lies intersect at an asterisk. In the literal mode of representation they are such as

$$(1)\ (aa)(ab),\quad (5)\ (ba)(aa),\quad (10)\ (ab)(ba),\quad (12)\ (ab)(bc),$$

i.e., such that the last letter in the first bracket is of the same sort as the first letter in the last bracket. We may add to these the pairs such as $(aa)(aa)$ which are really connecting pairs of two systems P_λ and P_μ. The pairs (2), (3), (4), (6), (7), (8), (11), may be called *unjoined* pairs.

318. Each joined pair is accompanied by a unit unique with respect to it, represented by the dot or asterisk in the same row as the dot or asterisk representing the first unit of the pair, and in the same column as the dot or asterisk representing the second unit of the pair. This unit may be taken as the product of the two units composing the pair. In the literal mode of representation we shall have

$$(ab)(bc)=(ac)$$

where a, b, c, or any two of them may be letters of the same sort.

319. The product of two unjoined pairs is taken to be a unit Z or 0, and if (ab) be any unit of P, we take

$$\begin{aligned}(ab)Z&=Z\\ Z(ab)&=Z\\ ZZ\ &=Z \text{ (sec. 308).}\end{aligned}$$

320. The units (ab), (ac), (bc), . . . , when thus dealt with as multipliers, multiplicands, and products, the products being those specified in the preceding sections, may be called *quadrates*.*

321. We may consider products of component collections of P or P_H (sec. 311); and we may regard these as unified, and thus arrive at various primitive algebras.

322. These quadrate algebras are all associative.

323. It will be convenient in many cases to write a quadrate (ab) thus $\frac{a}{b}$, assimilating it to the ordinary algebraic fraction, and a collection of quadrates (sec. 321) thus, $\frac{a}{b}+\frac{b}{c}+\frac{b}{d}$.

324. Quadrates have been arrived at by considering unified pairs in the case of single-heap systems. We might consider unified pairs in the case of systems of any other form, but in such case P would not have the characteristic form which we have been considering, and we should not get products. Thus if we considered discrete heaps, all the unified pairs would be distinguished from each other. We, of course, can and do consider, unified pairs of any system S, and can take their products as if they were

* See PIERCE on "Linear Associative Algebras," in the 'American Journal of Mathematics,' vol. iv., p. 215.

quadrates; but in such case we ignore differences, and do not deal with S, but with a single heap system of the same number of units (secs. 127, 128). The special form of S has its effect as the selector of special quadrates, or collections of quadrates (sec. 321).

325. Thus if S be a group the unified pairs divide up into systems, and we naturally select these. The product of two such systems of unified pairs is another such system. The unified systems (which are represented by the sorts of the pairs) give us a primitive associative algebra. The equations we obtain are precisely the same as those we should get if we considered chains in the manner indicated in section 234. In both instances the equations exhibit relations existing between sorts of letters, which in both instances represent simple networks. In the one instance we arrive at these relations by considering individual component pairs of simple networks, in the other by considering the whole systems of pairs composing those networks.

326. The unified aspects of any system S are a group. The unified pairs of these are substitutions by which any aspect of S is substituted for another. The product of any two of these is a substitution proper to S (sec. 105), viz., that which results from operating with the multiplier and multiplicand in succession.

327. A substitution proper to S is represented by a two-row constituent of the tabular representation of S. Each column of this constituent may be regarded as a quadrate, the whole constituent being thus regarded as a collection of quadrates. Thus a substitution may be represented either by a collection of quadrates obtained by taking pairs of S, or by a collection of quadrates obtained by taking pairs of the group of which the units are unified aspects of S.

Isolated Collections—Residuals—Satisfied Collections.

328. A collection of units which is such that each unit of the collection is unique with respect to the residue of the collection may be said to be *isolated*. Each unit of the collection may be termed the *residual* of the rest.

329. The residual of a pair a, b may be written (ab) (sec. 139), and similarly in the case of triads, &c.

330. We may graphically represent an isolated triad as in fig. 61, where λ is an

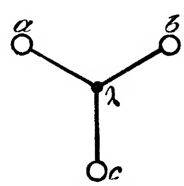


Fig. 61.

auxiliary graphical unit; a non-isolated triad being represented as in fig. 62, without links or auxiliary units. The same mode of representation may also be applied in the case of isolated n-ads where $n \neq 3$.

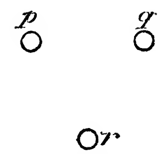


Fig. 62.

331. Where we are considering residuals of collections of r units, if a collection of units C is such that every component collection of r units has one and only one residual, which residual is also a component of the collection C, then C may be said to be *satisfied*.

Some Isolated-Triad Systems.—Family No. 1.

332. There are certain systems in which every component collection of two units has one residual and one only. One such system is met with in the case of the points of intersection of a plane cubic with coplanar straight lines; these make isolated triads of collinear points such that if a, b, c; d, e, f; g, h, i; a, d, g; b, e, h are isolated triads, c, f, i is one also; *i.e.*, we have $((ab)(de)) = ((ad)(be))$.

333. In another which I shall now consider the law of distribution of the isolated triads is such that if p, q, r, be a non-isolated triad; and if l be the residual of q, r; m of r, p; n of p, q; then l, m, n is an isolated triad; *i.e.*, we have $((pm)l) = (p(ml))$.

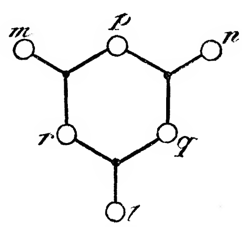


Fig. 63.

334. It is an immediate consequence of the law of distribution of sec. 333, that if s be the residual of l, p (fig. 63), it is also the residual of m, q and n, r. Thus we arrive at a collection of 7 units (comprising the isolated triads l, m, n; m, r, p; r, l, q;

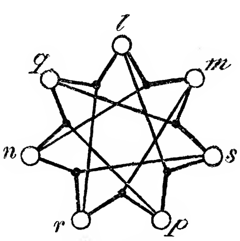


Fig. 64.

p, q, n; p, l, s; q, m, s; r, n, s), which contains the residual of each of its component pairs and is therefore satisfied. The whole collection is shown in fig. 64.

335. In a system of this species, if we add a unit to a satisfied collection C of m units, we must also add m others, getting $2m+1$ in all, before we can get a satisfied collection. This may be thus shown :—Let $a, b, c, d, \ldots$ be the units of C, and let a unit u be added to C, and let $a', b', c', \ldots$ be the residuals of u, a; u, b; u, c; .... respectively; then $a', b', c', \ldots$ must all be different units; for if a' and b' were identical we should have both u, a', a, and u, a', b, isolated triads. If then a unit u be added to C, m others $a', b', c', d', \ldots$ must also be added, making $2m+1$ in all. This collection of units will be satisfied, so that we need not add any more units. To prove this we have only to show that the residual of b', c', any two added units is a unit of the enlarged collection. Let the residual of b, c, which will be a unit of C be a, then we have the non-satisfied triad u, b, c, and the residuals of its pairs are b', c', and a which by sec. 333 form a satisfied, or isolated triad, for they are the same thing, so that the residual of b', c', is a.

336. Now every non-satisfied triad is a component of a satisfied collection of seven units; and by the preceding section if there be another unit added we get a collection of 15 units, if again another be added one of 31 units, and generally a satisfied collection consists of 2^n-1 units. Every system of the species considered is of course satisfied, and therefore contains 2^m-1 units.

337. If to any component collection C of one of these systems we add such residuals of the different component pairs as are not already in C, and repeat the process on the enlarged collection we get a definite satisfied component. The component R which consists of all the added units may be termed the *complement* of C. The complement of any pair is their residual. If C be itself satisfied there will be no complement. It does not follow if R be the complement of C, that C is the complement of R, though the complement of R must be a component of C. Thus the complement of p, q, n, s, l, in fig. 64 is m, r, but the complement of m, r is p.

338. Consider a system S of the family containing 2^n-1 units. Take any pair of S, add to it any unit of S not the residual, add to the resulting triad any unit of S not in its complement, add to the resulting tetrad any unit of S not in its complement, and so on. Proceeding in this way we get an n-ad, one of several, such that its complement comprises all the remaining units of S. The components of three, four, &c., units of the n-ad may be termed *principal* triads, tetrads, &c., of S. A principal n-ad has the same self-correspondences as a single heap of n units.

339. Let $a, b, c, d \ldots k, l$ be a principal component of a system S, substitute for a, b their residual γ, for γ and c their residual δ, for δ and d their residual ϵ, and so on until we get the pair λ, l, for which substitute their residual μ, which will be unique with respect to $a, b, c, d \ldots k, l$. We should arrive at μ if we took the units of the component in any other order. To show this it is only necessary to prove it for an interchange of any two units, for by a succession of such interchanges we can pass to any fresh order. Let us then interchange any two units c and d, and let ξ be the residual of γ, d, then we have to show that ϵ is the residual of ξ, c. This

follows at once from the law of sec. 333, for γ, δ, d is a non-isolated triad and γ, δ, c, δ, d, ϵ, and d, γ, ξ are isolated; thus c, ϵ, ξ is isolated and ϵ is the residual of ξ, c. We should arrive at μ if we were at any stage to substitute the residual of *any* two units of the reduced component, instead of always taking the last substituted residual as one of the two; *e.g.*, after substituting δ for γ, c, we might substitute the residual of h, k for h, k. The unit μ is clearly a component of the complement of $a, b, c, d \ldots . k, l$. It is obvious that if we started with the component $a, b, c, d \ldots . k, \mu$ we should arrive at the unit l, and generally any other unit of $a, b, c, d \ldots . k, l, \mu$ would be arrived at from the rest. In fact the whole collection $a, b, c, d \ldots . k, l, \mu$ is isolated, and each unit is thus a residual of the remainder, the only one that there is.

340. The process of the preceding section may be applied to any component whatever of S, with this modification in the case of non-principal components, viz., that where the residual of any pair is included in the component so that we cannot add it to the latter in substitution for the pair, in such case the whole triad must be removed from the component. This amounts to removing any isolated r-ad which is contained in the component. If in the result one unit remains, it will be the only residual; if no unit remains the component is isolated.

341. We may classify the units of S relatively to a principal component of n units thus, viz., we have

(1) The principal component of	n	units
(2) The residuals of the pairs of these	$\dfrac{\lfloor n}{\lfloor 2 \lfloor n-2}$	in number
(3) ,, ,, triads	$\dfrac{\lfloor n}{\lfloor 3 \lfloor n-3}$	,,
(4) ,, ,, tetrads	$\dfrac{\lfloor n}{\lfloor 4 \lfloor n-4}$	,,
. . . .		
(r) ,, ,, r-ads	$\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$	,,
. . . .		
(n) The residual of the n units	1	,,
Total	2^n-1	units.

342. If we represent the units of the principal component (1) by the letters a, b, c, ... , we may represent every other unit of S as a residual of two or more of these in the manner indicated in sec. 329. The order of the letters in the brackets will be immaterial. The symbol representing the residual of any two units of S will be arrived at by aggregating the letters contained in one or other of the two symbols

representing the two units, but not in both; *e.g.*, the product of $(abcd)$ and (cde) will be (abe).

343. If we call the residual (ab) the product of a and b, and add to S a system of one unit 1 such that

$$(aa)=1$$
$$(a1)=a$$

whatever unit of S a may be (*cf.* sec. 308), we have an algebra subject to the associative law, viz., we have

$$((ab).c)=(a.(bc))$$

Some Isolated-Triad Systems.—Family No. 2.

344. Another distribution of isolated triads which may exist is this:—If a, b, c, be an isolated triad, and if o, a, l; o, b, m; o, c, n are also isolated triads, then l, m, n is an isolated triad. Thus $((oa)(ob))=(oc)=(o(ab))$.

345. Here, if a, b, c, d . . . . be any satisfied component of the system, and a unit o be added, the residuals (oa), (ob), (oc) . . . . will also compose a satisfied collection, the triad (oa), (ob), (oc), being isolated or not according as a, b, c is an isolated triad, or not.

346. The various residuals $(a(oa))$ or o, $(a(ob))$, $(a(oc))$ . . . . are all distinct units, no two of these symbols denoting the same unit; and they constitute a satisfied collection, the triad $(a(ob))$, $(a(oc))$, $(a(od))$ being isolated or not according as (ob), (oc), (od) is isolated or not, *i.e.*, according as b, c, d, is isolated or not; and similarly in the case of the residuals $(b(oa))$, $(b(ob))$ . . . . .

347. We have $(a(ob)) = ((ao)(ab))$ (sec. 344), and thus the united collections

a, b, c . . . . .

(oa), (ob), (oc) . . . .

$(a(oa))$, $(b(oa))$, $(c(oa))$ . . . . .

constitute a satisfied collection. If the number of units in the collection a, b, c, ... is n, the total number in the satisfied collection arrived at by adding o is $3n$.

348. Thus the total number of units in any satisfied collection of the species now under consideration is 3^m.

349. If we call the residual (ab) the *product* of a and b, and write it for simplicity ab, we see that $ab.c \neq a.bc$, so that the associative law does not hold, as in the preceding systems. But we have $a.bc = ab.ac$, so that the algebra is self-distributive. We can have, as in the case of the preceding family of systems,

$$aa=1$$
$$a1=1.$$

Geometry in General.

350. In most geometrical investigations the units constitute a system of a high order of multiplicity; we have points, straight lines, conics, cubics, &c., unified collections of two, three, or more of these, operators such as quarternions, &c., &c., &c. It will, however, be sufficient for the purpose of illustration to refer briefly to some comparatively simple systems.

351. It is to be understood that points and the line at infinity are not regarded as distinguished from other points and lines, and consequently parallelism is looked upon merely as intersection on a particular line.

System of Coplanar Points and Straight Lines.

352. In a general consideration of coplanar points and straight lines the points compose a single system, being undistinguished from each other, and the same thing is the case with the straight lines. The connecting pairs of the two systems are of two sorts, for a point may either lie on a line or off it; and, as there is no other special relation between a point and a line, the system of connecting pairs is a double one.

353. We may graphically represent the lines by the graphical units ○ ○ ○ ○, and the points by the graphical units • • • • •. If a point lies on a line we may connect the corresponding graphical units by a link thus ○——•, no link being drawn in the other case.

354. The number of points on each line is infinite, and an infinite number of lines pass through each point. Every two points lie on one line and one only, so that if a and b are any two points, the line P which is linked to both is unique with respect to a, b (fig. 65). Pairs of points are accordingly all undistinguished from each other.

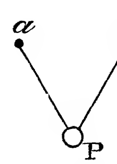


Fig. 65.

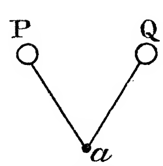


Fig. 66.

355. Every two lines pass through one common point, and one only; so that if there is a point a linked to two lines P and Q, a is unique with respect to PQ (fig. 66).

356. Every form which component collections of lines and points possess is also possessed by component collections of points and lines; the two component single systems being of precisely the same form.

357. We may take as fundamental laws defining the distribution of the links, and therefore the form of the double system the two well-known theorems:

(1.) If the three points in each of the nine triads of points

$a, b, c;\ \ a, d, e;\ \ a, g, f;\ \ b, d, h;\ \ c, e, h;\ \ d, f, l;\ \ e, g, l;\ \ b, f, k;\ \ c, g, k.$

are collinear, the points of the triad h, k, l are also collinear (fig. 67).

(2.) If the three points in each of the eight triads of points

$a, b, c;\ \ a, h, f;\ \ d, e, f;\ \ a, g, e;\ \ b, g, d;\ \ c, h, d;\ \ b, k, f;\ \ c, k, e$

are collinear, the points in the triad g, h, k are also collinear (fig. 68).

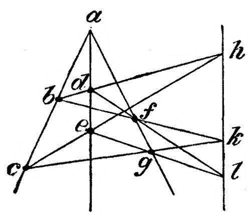


Fig. 67.

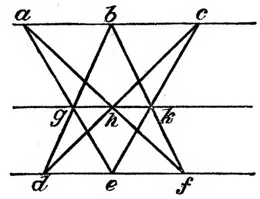


Fig. 68.

Coplanar Points, Lines, and Conics.

358. In the case of the treble system composed of coplanar points, straight lines, and conics, each line touches, cuts, or does not touch or cut each conic, and each point lies on, in, or outside each conic; we thus have three sorts of connecting pairs in each case. We may conveniently connect the graphical unit representing a conic by links to the graphical units representing points lying on the conic, and to the graphical units representing lines touching the conic; no links being drawn in the other cases; so that the unlinked pairs are a double system.

359. If a be any point, C any conic, of the system, there is one line and one line only of the system which is unique with respect to the pair a, C, viz., the polar λ of a with respect to C. Likewise a is the only point unique with respect to λ, C.

Logic.

360. It will be convenient to approach the consideration of Formal Logic from the standpoint of the algebraist. Classes are denoted by terms A, B, C, D, . . . The class which just contains, and the class which is just contained in, all the classes A, B, C, are denoted respectively by the sum A+B+C, and the product ABC of the terms denoting the several classes. Whatever class A may be, we have AA=A, and A+A=A. Addition and multiplication are each commutative and associative; and are also distributive as regards each other, so that we have

$$(A+B)(C+D)=AC+AD+BC+BD$$
$$AB+CD=(A+C)(A+D)(B+C)(B+D)$$

If AB=A, which implies also A+B=B, then A is contained in B.

361. Two classes U and Z are considered which are such that whatever class A may be, we have

$$\mathrm{U}+\mathrm{A}=\mathrm{U}, \text{ and therefore also } \mathrm{UA}=\mathrm{A}$$
$$\mathrm{Z}+\mathrm{A}=\mathrm{A}, \text{ and therefore also } \mathrm{ZA}=\mathrm{Z}$$

so that U contains every class, and Z is contained in every class. U is the class called the "universe," Z is that called the "non-existent" class. The class U need not be taken as the actual universe of thought, but as that class which contains all classes considered in an investigation. So Z may be taken to be that class which in any investigation is contained in all others considered, and is ignored, or regarded as having the quality of "non-existence."

362. Every class A is accompanied by a class A′ which may be called its *obverse*, and is such that

$$\mathrm{A}+\mathrm{A}'=\mathrm{U}$$
$$\mathrm{AA}'=\mathrm{Z}$$

so that A is the obverse of A′. A′ is "not A." U and Z are obverses of each other. We have

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\ldots)'=\mathrm{A}'\mathrm{B}'\mathrm{C}'\mathrm{D}'\ldots$$
$$(\mathrm{ABCD}\ldots)'=\mathrm{A}'+\mathrm{B}'+\mathrm{C}'+\mathrm{D}'+\ldots$$

363. Now, consider a collection of n classes a, b, c, d, . . . such that the product of every two is a class z; *i.e.*, such that no two classes of the collection have any part in common except that which is common to all the classes, and may therefore, when our attention is confined to the classes of the collection and their aggregates, be regarded as immaterial, and therefore be treated as non-existent. Such a collection may be said to consist of *separated* classes. Taking these, their sums two, three, four, &c. together, and the class z, we have a collection of 2^n classes, which may be called a *full* set, and may be said to be *derived* from the collection of n separated classes.* All the classes of the set are contained in a class u of the set which is the sum of the separated classes from which the full set is derived, and may be called the *universe* of the set. The class z may be called the *zero* of the set. The whole system of classes involved in any inquiry is a system of 2^m classes where m is the number of separated classes, the product of each two of these being the non-existent class Z. In discussing a system of classes we have under consideration a number of full sets with their accompanying universes and zero classes.

364. If a be any class of a full set of which u is the universe and z the zero class, we have

$$a+u=u \text{ and } au=a$$
$$a+z=a \text{ and } az=z$$

* A more general meaning is given to the expression "full set" in sec. 381, and it must not therefore be assumed that conversely every full set can be derived in the manner just indicated.

and a is accompanied by a class $\bar{a}$ such that

$$a+\bar{a}=u$$
$$a\bar{a}=z.$$

Also we have

$$(\overline{a+b+c+d+\ldots})=\bar{a}\ \bar{b}\ \bar{c}\ \bar{d}\ldots$$
$$(\overline{abcd\ldots})=\bar{a}+\bar{b}+\bar{c}+\bar{d}+\ldots$$

where $a, b, c, d, \ldots$ are any classes of the full set. Thus the relations of the component classes of a full set are similar to those of the component classes of a whole system. The class $\bar{a}$ may be called the "*obverse of a in the full set.*" It is not of course the true obverse of a, *i.e.*, is not a', unless the full set is the whole system.

365. Any collection of n classes $a, b, c, d, \ldots$ which are such that they and their various sums and products are all different classes may be said to be a collection of *unrestricted* classes. The minimum full set of which a collection of n unrestricted classes is a component contains 2^{2^n} classes; for the separated classes of such full set are $abcd\ldots$, $a'bcd\ldots$, $a'b'cd\ldots$, $a'bc'd\ldots$, $a'b'c'd\ldots$, $a'b'c'd'\ldots$, &c., which are 2^n in number. This full set may be said to be *derived* from the collection of n unrestricted classes.

366. If $a, b, c, d, \ldots$ be any collection of unrestricted classes, we can always find a class λ of the derived full set, such that the classes $\lambda a, \lambda b, \lambda c, \lambda d, \ldots$ have any desired class relations to each other.*

367. Thus if we have a collection of 2^n unrestricted classes $a, b, c, d, \ldots$, there will be a class λ such that $\lambda a, \lambda b, \lambda c, \lambda d, \ldots$ constitute a full set, and another class μ such that $\mu a, \mu b, \mu c, \mu d, \ldots$ are a collection of separated classes.

368. Suppose we have under consideration a full set T of 2^n classes. We may, if we please, ignore the relations of inclusion, &c., which they have to each other, and treating them as a collection of separated classes constitute a derived full set P of 2^{2^n} classes containing them, their aggregates, and a zero class. Here the units of T are the products $\tau a, \tau b, \tau c, \tau d, \ldots$ of a class τ and a collection $a, b, c, d, \ldots$ of 2^n unrestricted classes. When we ignore the relations of inclusion, &c., which $\tau a, \tau b, \tau c, \tau d, \ldots$ have to each other, we no longer deal with $\tau a, \tau b, \tau c, \tau d, \ldots$ but with a collection $\pi a, \pi b, \pi c, \pi d, \ldots$ of separated classes.

369. The obverse in T (sec. 364) of any class τa is $\tau a'$, the class containing all classes of T which have only the zero class in common with τa. The obverse in P of πa is $\pi a'$, the class which contains all the separated classes $\pi b, \pi c, \pi d, \ldots$; viz., all those classes arrived at by ignoring the relations of inclusion, &c., of $\tau b, \tau c, \tau d, \ldots$

* Thus Mr. VENN in his 'Symbolic Logic,' at chap. v., employs intersecting ellipses to represent unrestricted classes. He then regards parts of these as eliminated, and so makes the ellipses represent classes having any desired relations. Here λ represents the class composed of the uneliminated spaces.

370. When any collection ξa, ξb, ξc, ξd, . . . of classes of a full set T have certain relations of inclusion, &c., to each other; instead of saying that ξa, ξb, ξc, ξd, . . . have these relations, we may say that a, b, c, d, . . . have the relations "in ξ." A class ξ may be completely defined by stating that it is the most extensive class in which certain relations hold between other classes. Thus the class $\bar{a}+b$ of a full set in which a and b are unrestricted is fully defined by saying that it is the most extensive class of the set in which "all a is b," or in which $ab=a$. Further, instead of using this definition, we may simply employ the equation $ab=a$ without other words; it will then assert that the class $\bar{a}+b$ is that under consideration and will be a mark or term denoting that class. The equation $a=u$ will be a term denoting the class a of the set.

371. We might use $ab=a$ to denote, not the class $a+b$, but the class which contains all those classes arrived at by ignoring the relations of inclusion, &c., between $\bar{a}+b$ and the various classes contained in $\bar{a}+b$.* It is in this sense that such a mark is usually employed.

372. Taking $ab=a$ as used in the last paragraph, the term $ab\neq a$, *i.e.*, "some a only is b," will represent the class arrived at by ignoring the relations of inclusion, &c., between all those classes of T which are not included in $ab=a$. Here $ab\neq a$ is the obverse of $ab=a$ in P the full set derived from the full set T by regarding the classes of the latter as separated.

373. Terms such as $ab=a$, $ab\neq a$, are called *propositions*. The difference between propositions and other terms is accidental and not essential matter of exact thought; propositions, like other terms, merely denote classes.

374. When we consider a full set T of 2^n classes τa, τb, τc, τd, . . . , and the related propositional classes, we consider a system S of $2^{2^{2^n}}$ classes, viz., that derived from the 2^n unrestricted classes a, b, c, d, . . . ; the classes τ and π being other classes of S. If as is generally the case T is regarded as derived from n unrestricted classes, represented by single letters A, B, C, D, . . . , the number of classes in T will be 2^{2^n}, in P will be $2^{2^{2^n}}$, in S will be $2^{2^{2^{2^n}}}$.

375. In a syllogism we have two classes called the *premisses*, *e.g.*, (1) $(ab=a)$, (2) $(bc=b)$. We consider the class which is the product of these, viz., $(ab=a)\,(bc=b)$, *i.e.*, that which contains all the separated classes of T, which both (1) and (2) do. This class includes some classes, is included in others, &c. It is included in $(ac=a)$. The syllogism indicates this fact, the class $(ac=a)$ being called the *conclusion*.

376. We have then under discussion in an investigation of classes a system of 2^m units consisting of

* If the proposition "all a is b" is taken to imply the existence of its subject a, we must exclude the zero class of T. Similarly in the case of the proposition "some a only is b" (sec. 372).

(0) a class Z,

(1) m separated classes each containing Z,

(2) $\frac{m(m-1)}{2}$ classes each containing two separated classes,

. . . .

(r) $\frac{\lfloor m}{\lfloor m-r \lfloor r}$ classes each containing r separated classes,

. . . .

(m) a class U containing all the separated classes.

Each class of the collection (r) may be derived by adding to some class s of the collection ($r-1$) a class of (1) not contained in s.

377. We have apparently as an essential accompaniment of the idea of classes the notion of the inclusion of one class in another. Inclusion is usually regarded as a relation between *two* classes. This is not, however, really so. The notion of inclusion essentially involves that of a chain of classes with two terminals, viz., a class U which contains all classes, and a class Z which is contained in all classes, of the chain. In dealing with a class A we really deal also with the classes U and Z; and in dealing with two classes A and B we deal with the four classes A, B, U, Z. When this fact is recognised, inclusion, as commonly conceived, is seen to be a relation which, as far as the processes of exact thought are concerned, is accidental; that which is essential in it depending upon the places occupied in a system of classes by two classes relatively to two others (sec. 388).

378. The classes U and Z are not, apart from accidents, distinguished from any other classes. The system of which the units are classes is a single one. The reason why U and Z seem to hold exceptional positions is that when we discuss classes we consider their positions with reference to U and Z, which, being constantly under consideration, acquire an accidental importance.

379. There are two distinct sorts of inclusion, viz., *direct* and *indirect*. If a is contained in b, and b in c, then the inclusion of a in c may be said to be *indirect;* if a is contained in c, and there is no class b under consideration such that a is contained in b and b in c, then the inclusion of a in c may be said to be *direct*. The two classes a and c may in the latter case be said to be *adjacent*, in the former *non-adjacent*. The relations of adjacency and non-adjacency are relations between *two* classes, and do not depend upon the consideration of the classes U and Z. In a system of 2^m classes every class is adjacent to m classes. Thus in the case of the collections of sec. 376 every class of the collection (r) is adjacent to r classes of the collection ($r-1$) and $m-r$ of the collection ($r+1$).

380. The form of a system of 2^m classes is completely defined by the division of the pairs of classes into adjacent and non-adjacent pairs. It may therefore be graphically

represented by 2^m graphical units, of which pairs representing pairs of adjacent classes are linked, and pairs representing pairs of non-adjacent classes are not linked.

381. A convenient mode of building up a graphical representation of such a system is the following:—Suppose we have a graphical representation of any collection of units, say a *diagram*. To this we can add a precisely similar diagram, say we can *repeat* the diagram. The two diagrams will correspond unit to unit in one or more ways. Taking one of these ways only, we can link each graphical unit of one diagram to the corresponding unit of the other, say we can *connect* the two diagrams. Now draw a single graphical unit, repeat it, and connect the two diagrams (fig. 69). Repeat the resulting diagram and connect the two diagrams (fig. 70). Repeat the resulting diagram and connect the two diagrams (fig. 71). Repeat the resulting diagram and connect the two diagrams (fig. 72). This process can be carried on indefinitely, the number of units being each time doubled, so that the number of units after n processes is 2^n.

Fig. 69.

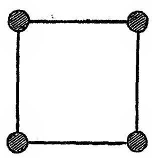


Fig. 70.

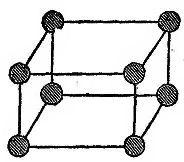


Fig. 71.

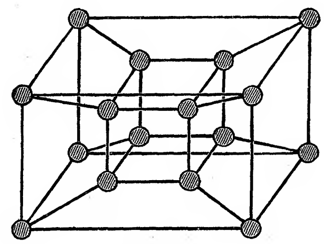


Fig. 72.

382. Each of the diagrams may be said to represent a *full set* of the whole system finally arrived at. There are of course other full sets besides these in the system. Thus in fig. 72 there are 8 full sets represented, such as that in fig. 71; and generally in any system of 2^m units there are $2m$ full sets of 2^{m-1} units.

383. We can pass from one graphical unit to another along links. We can do this by passing along a minimum number of links, or by more circuitous routes. The number of links passed over in the case in which we pass over a minimum number may be termed the *distance* between the two units. Adjacent units are at a distance 1 apart.

384. In the following sections it is to be understood that when chains connecting two units are spoken of, those containing the same number of links as the two units are distant apart are referred to. We can, of course, by zigzagging about, make the chains much longer, but such chains are not those considered, which are only the chains of minimum length.

385. Let σ_0 be any unit of a class system of 2^m units. There are m units linked to σ_0, composing a collection which may be called σ_1, every unit of which is at a distance 1 from σ_0. We have also a collection σ_2 of $\frac{m(m-1)}{2}$ units other than σ_0 and σ_1, each of the units of which is at a distance 2 from σ_0. So, generally, we have

a collection σ_r of $\frac{\lfloor m}{\lfloor r \lfloor m-r}$ units, each of which is at a distance r from σ_0. When $r=m$ we get a single unit σ_m which is the obverse of σ_0, and is unique with respect to σ_0, being the only unit of the system which is so. The units of any collection σ_r are each linked to m units of σ_{r-1} and to $m-r$ of σ_{r+1}, but to no other units of the system, so that none of them are linked to each other. We see here that the relation of a class σ_0 to its obverse σ_m is one not depending on the relations of σ_0 and σ_m to U and Z, though expressible in terms of such relations.

386. Consider now two collections of chains of links, (1) consisting of chains from σ_0 to a unit α, (2) consisting of chains from σ_0 to a unit β (see sec. 384). There are some graphical units through which chains of both collections pass. If we classify these according to their distances from σ_0, we find that the number in each class is always greater than 1, except in the case of the greatest distance, when there is one only, which is therefore unique with respect to σ_0, α, β. If the chain from σ_0 to α passes through β, it is clear that the unit is β itself. In the same way, if we take any number of units $\alpha, \beta, \gamma, \delta, \ldots$, there is a unit unique with respect to $\sigma_0, \alpha, \beta, \gamma, \delta$, which may be one of the units $\alpha, \beta, \gamma, \delta, \ldots$ We may obtain a like unit unique with respect to $\sigma_m, \alpha, \beta, \gamma, \delta, \ldots$ We may term the unit in the former case the product of $\alpha, \beta, \gamma, \delta, \ldots$ with respect to σ_0; and in the latter case the product with respect to σ_m.

387. If σ_0 be Z, then the product of any units with reference to σ_0 will be what has been spoken of in sec. 360, simply as the *product* of the classes represented by those units. The product with respect to σ_m, which will be U, will in such case be that which was defined in sec. 360 as the *sum* of the classes.

388. If a chain from U to Z passes through two units α and β, first through α and then through β, then α and β represent classes which are such that α contains β. If no chain from U to Z passes through both α and β, then neither class is contained in the other. This makes it clear that the relation of inclusion is one in essentials depending merely on the form of a system of classes, and the position two units occupy in it relatively to two others (U and Z); or rather, as U and Z are unique with respect to each other, to one other, viz., either U or Z.

389. Suppose we have a system S containing the n unrestricted classes $a, b, c, d, \ldots$, the obverses U and Z, and the collection C of separated classes $abcd \ldots$, $a'bcd \ldots$, $ab'c'd \ldots$, &c. Here C has the same self-correspondences as a single heap. Now, if we confine our attention to those self-correspondences of S in which the collection of $2n$ units consisting of $a, b, c, d, \ldots$, and their obverses is self-correspondent, the collection C will have those self-correspondences only which are characteristic of a full set; and will not have any correspondences with other collections; also the other units of S (except Z) which may then be regarded as aggregates of the units of C taken two, three, &c., together will correspond only if they are aggregates of component collections of C which correspond; *i.e.*, they and C will have correspondences as if they

composed as many systems as there are *types* of component collections of a full set of 2^n classes.

390. If $n=4$, so that C has 16 component units, the number of these types will be 397, so that a class system containing 16 classes has 397 different sorts of component collections.* It must be noted that it is distinguished *collections*, and not distinguished *aspects* of collections, with which we are here dealing.

391. The number of types in a system of 4 units is 5. In one of 8 units it is 21. In both cases components containing all units of the systems are included.

* See Professor CLIFFORD "On the Types of Compound Statement involving Four Classes," in the 'Proceedings of the Manchester Philosophical Society,' vol. vi., Third Series. In this paper Professor CLIFFORD denotes the 2^{16} classes of a system derived from four unrestricted classes A, B, C, D, by statements or propositions. He regards the classes A, B, C, D, and their obverses a, b, c, d, as distinguished from all other classes of the system, and thus regards the separated classes as constituting a full system, and consequently gets the division of all the classes into 396 sorts, not including the aggregate of all the classes. In the view taken in this memoir, the 2^{16} classes should all be regarded as undistinguished from each other, the distinction raised between the classes A, B, C, D, a, b, c, d, and others, being due to the accident of particular classes being selected to be denoted by single letters, and not to any essential differences. This does not, of course, affect the conclusion that the number of distinct types of component collections of a system of 16 classes is, if we do not include the collection which is the aggregate of all the classes, 396, or if we do include such collection, 397.